

# CONTROL I

ELEN3016

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## Time Domain Analysis

(Lecture 5)

# Overview

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- First Things First!
- Time Domain Analysis (**Overview**)
- Examples
- Tutorial Exercises & Homework
- **Next Attraction!**

# First Things First!

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- Lab Notes

- To be finalised during next week.

- Misprints

- Burns: p 52; Fig 3.18, p 53; p 54; Eq (3.68), p 56; p 57.

# Time-Domain Analysis

- Linear System Output (from Signals & Systems)

$$x_o(t) = \underbrace{\text{Zero - Input Response}}_{\text{Due to initial conditions}} + \underbrace{\overbrace{\text{Zero - State Response}}^{\text{Transient Response} + \text{Forced Response}}}_{\text{Due to convolution of input \& impulse response}} \quad (\text{Lathi, p 53})$$

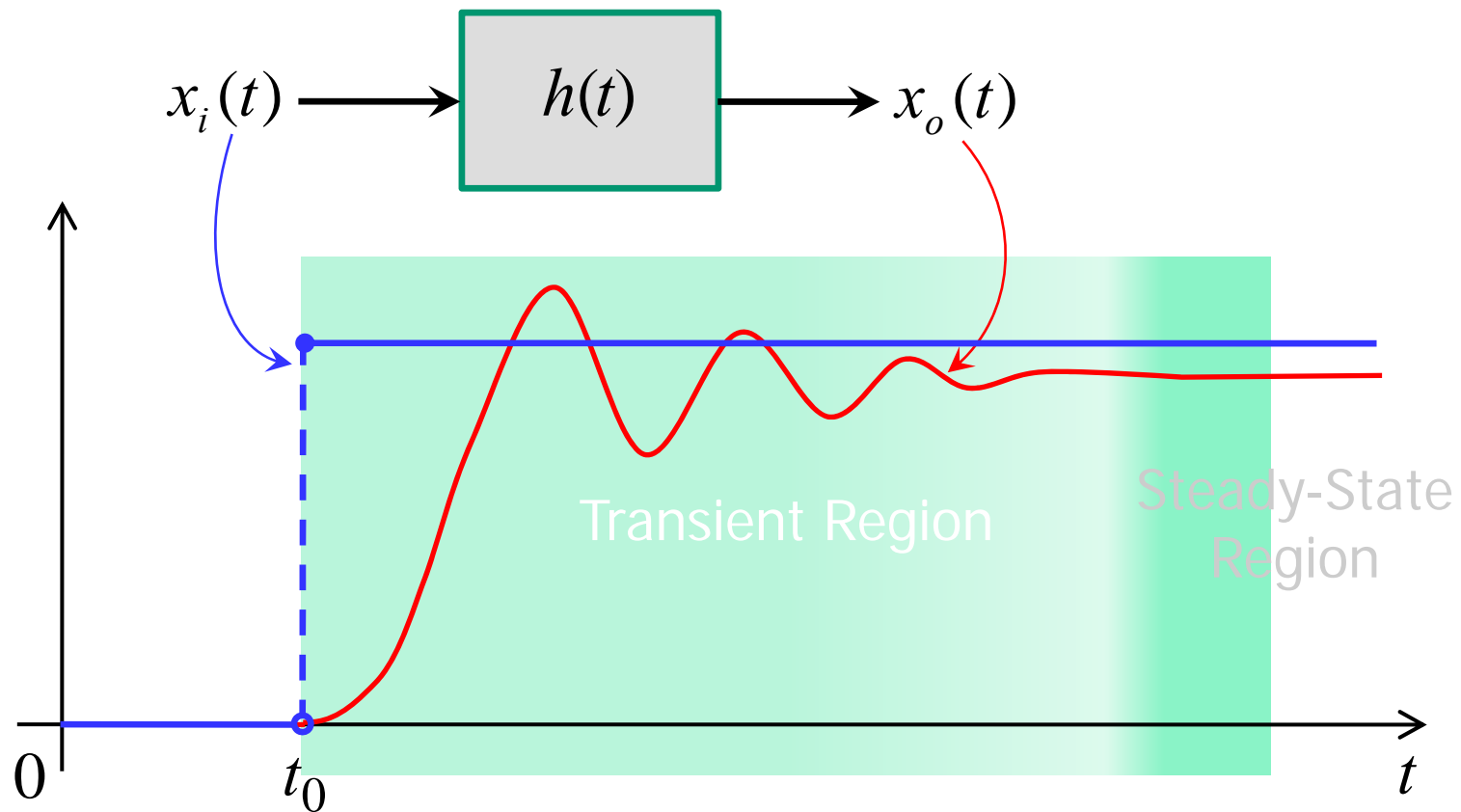
$$= \text{Natural Response} + \text{Forced Response} \quad (\text{Lathi, p 80})$$

$$= \text{Transient Response} + \text{Steady-State Response} \quad (\text{Lathi, p 82})$$

- For Sinusoidal Inputs
  - Transient Response = Natural Response
  - Steady-State Response = Forced Response

# Time-Domain Analysis

- Transient & Steady-State Regions



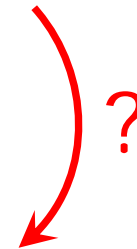
# Time-Domain Analysis

- Prototype 2<sup>nd</sup>-Order Transfer Function

$$a\ddot{x}_o(t) + b\dot{x}_o(t) + cx_o(t) = ex_i(t)$$

$$G(s) = \frac{X_o(s)}{X_i(s)} = \frac{e}{as^2 + bs + c}$$

$$= \frac{K\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$



# Time-Domain Analysis

- Prototype 2<sup>nd</sup>-Order Transfer Function

$$a\ddot{x}_o(t) + b\dot{x}_o(t) + cx_o(t) = ex_i(t)$$

$$\begin{aligned} G(s) &= \frac{X_o(s)}{X_i(s)} = \frac{e}{as^2 + bs + c} = \frac{\frac{e}{a}}{s^2 + \frac{b}{a}s + \frac{c}{a}} \\ &= \frac{K\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} \end{aligned}$$

# Time-Domain Analysis

- Prototype 2<sup>nd</sup>-Order Transfer Function

$$K = \frac{e}{c} \quad \text{Steady - state gain}$$

$$\omega_n = \sqrt{\frac{c}{a}} \quad \text{Natural frequency}$$

$$\zeta = \frac{1}{2} \frac{b}{\sqrt{ac}} \quad \text{Damping ratio}$$

$$\omega_n \zeta = \frac{b}{2a} \quad \text{Damping factor}$$



# Time-Domain Analysis

- Poles of the Prototype 2<sup>nd</sup>-Order System

- Over-damped ( $b^2 > 4ac$  or  $\zeta > 1$ ):

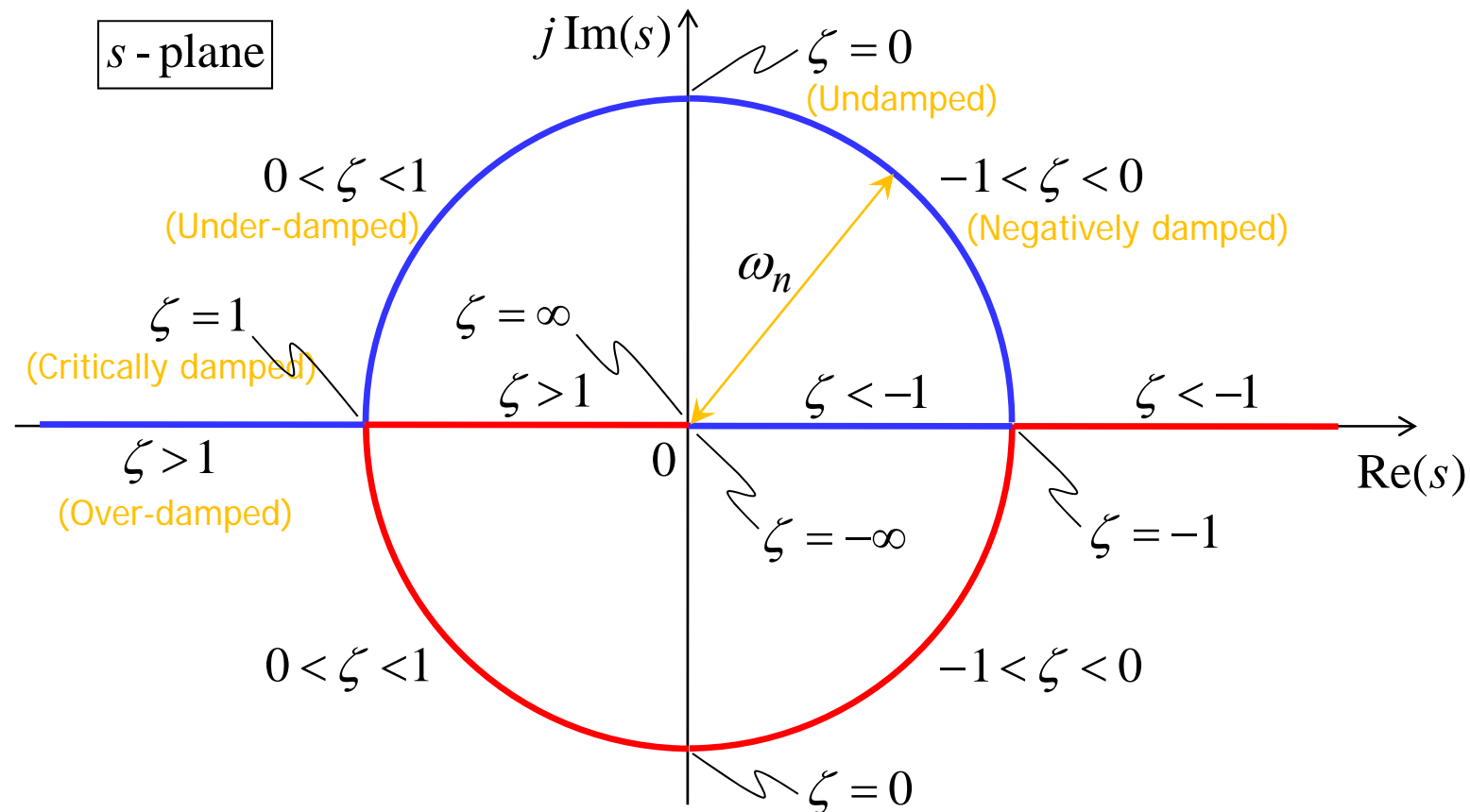
$$s_1, s_2 = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = -\omega_n \zeta \pm \omega_n \sqrt{\zeta^2 - 1}$$

- Under-damped ( $b^2 < 4ac$  or  $\zeta < 1$ ):

$$s_1, s_2 = \frac{-b \pm j\sqrt{4ac - b^2}}{2a} = -\omega_n \zeta \pm j \overbrace{\omega_n \sqrt{1 - \zeta^2}}^{\text{Damped frequency, } \omega_d}$$

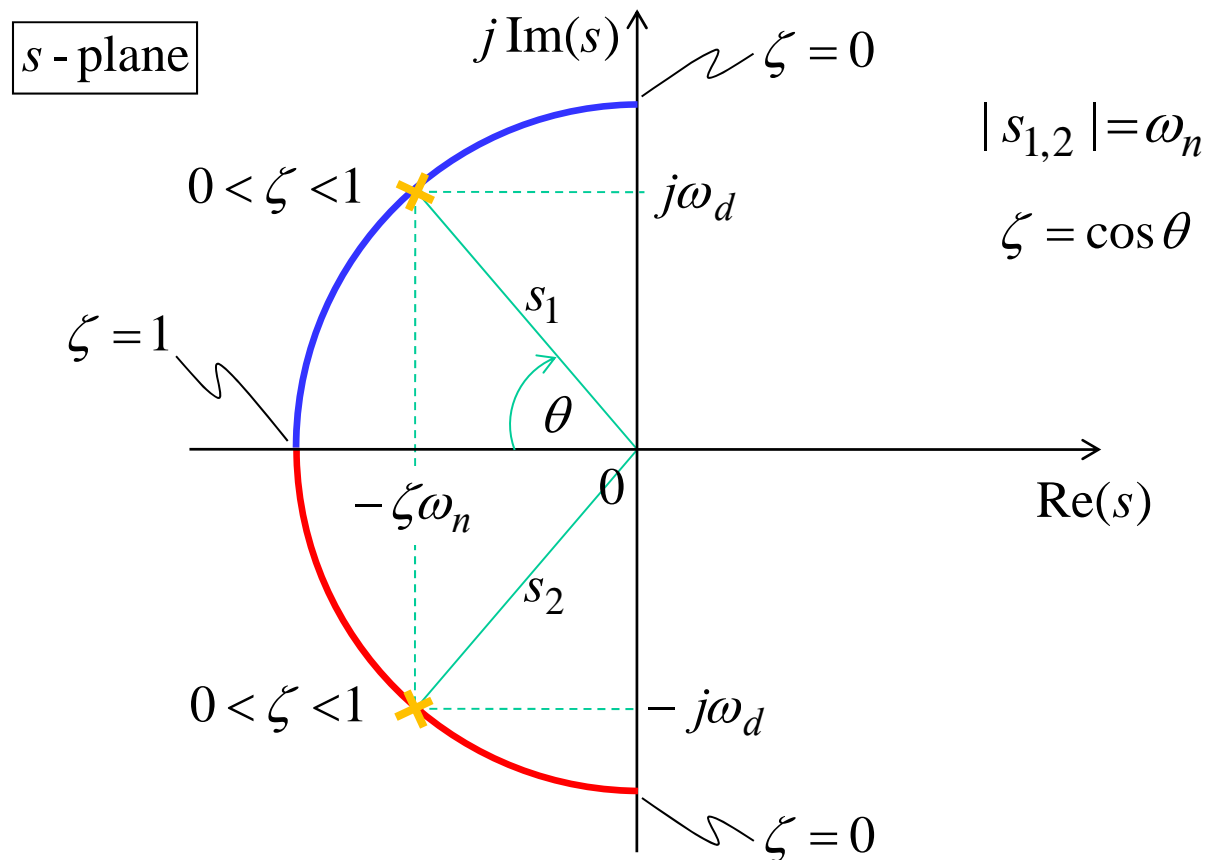
# Time-Domain Analysis

- Poles of the Prototype 2<sup>nd</sup>-Order System



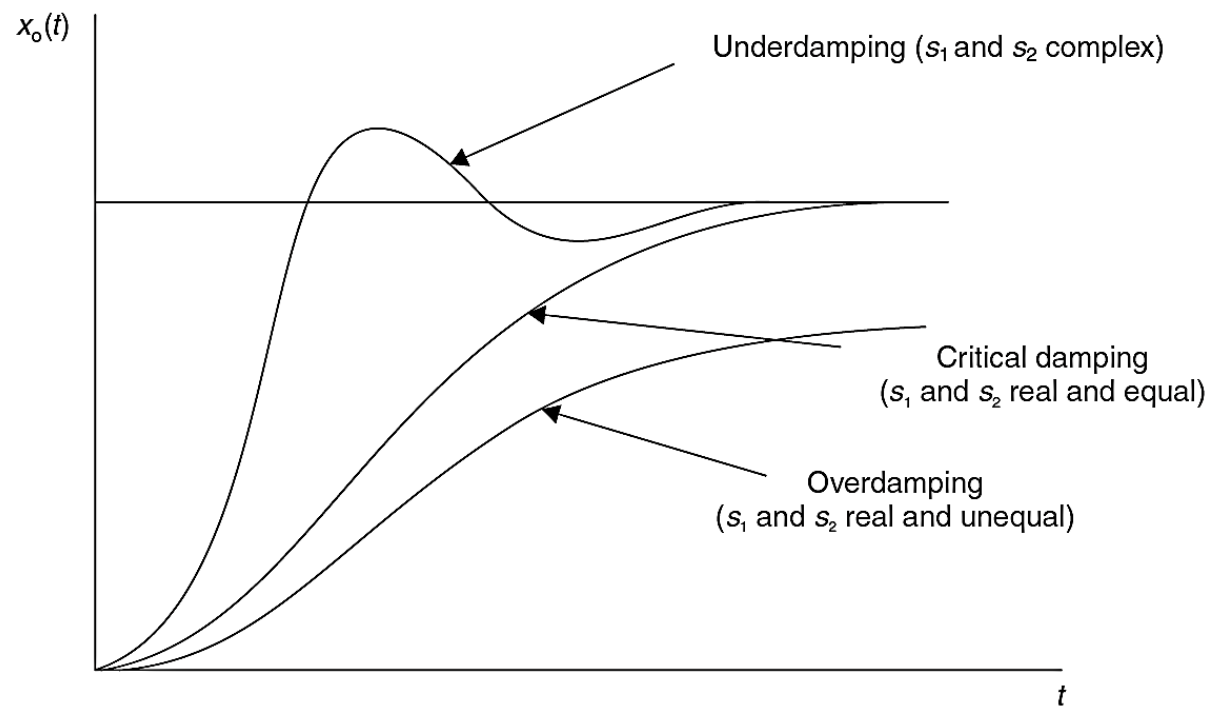
# Time-Domain Analysis

- Poles of the Prototype 2<sup>nd</sup>-Order System



# Time-Domain Analysis

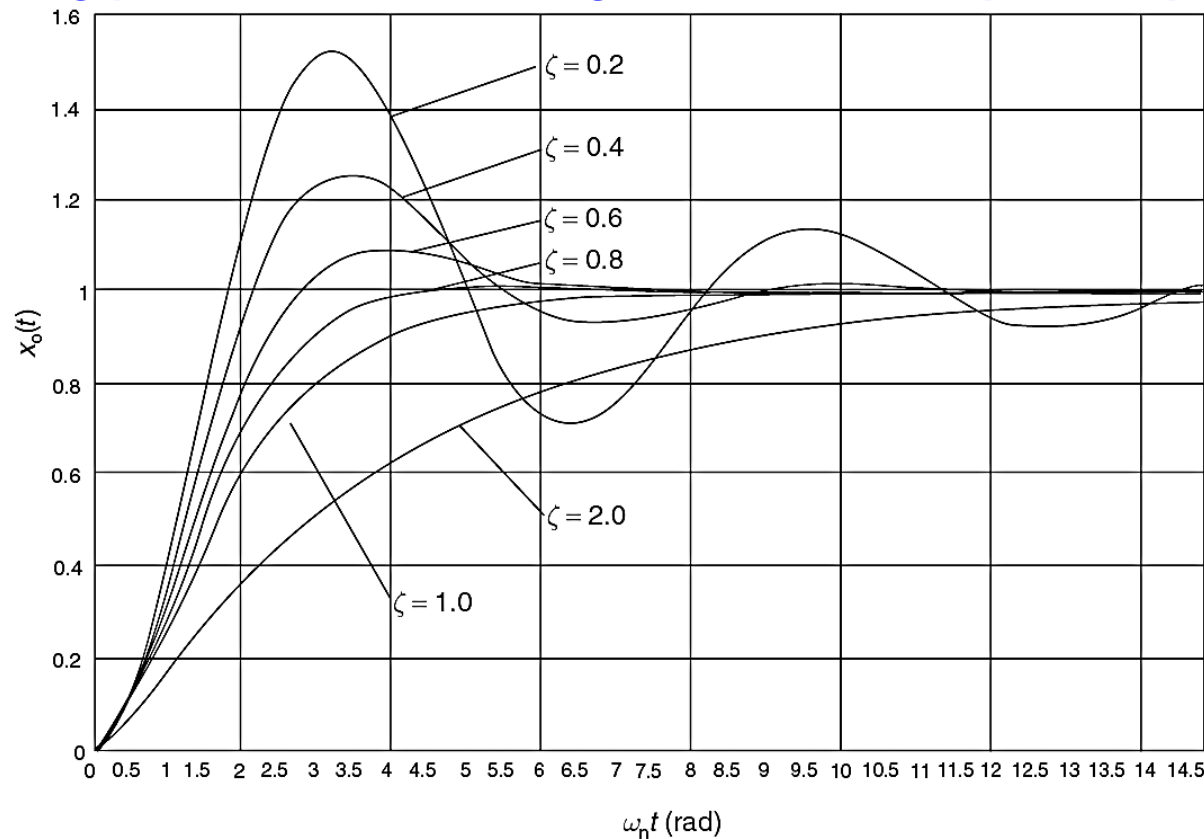
- Prototype 2<sup>nd</sup>-Order System – Step Response



**Fig. 3.16** Effect that roots of the characteristic equation have on the damping of a second-order system.

# Time-Domain Analysis

- Prototype 2<sup>nd</sup>-Order System – Step Response



**Fig. 3.19** Unit step response of a second-order system.

# Time-Domain Analysis

- Prototype 2<sup>nd</sup>-Order System – Step Response
  - Refer to Burns, Figures 3.16 & 3.19.
  - **Observation 1:**  $\omega_d$  increases (and approaches  $\omega_n$ ) as  $\zeta$  approaches zero.
  - **Observation 2:** Output settles in the shortest time when  $\zeta = 1$ .
  - Study the derivations in Burns, Section 3.6.4.

# Time-Domain Analysis

- 2<sup>nd</sup>-Order System – Step Response Analysis

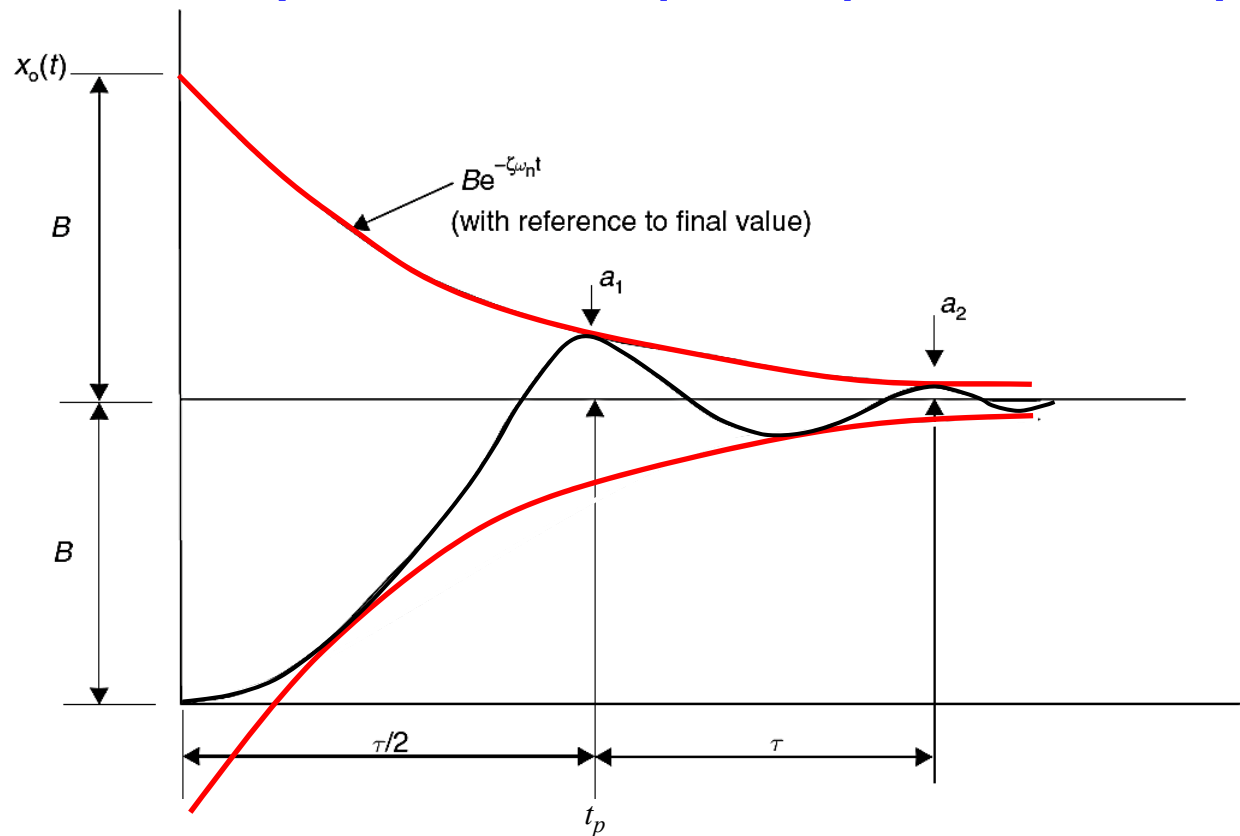


Fig. 3.20 Step response analysis.

# Time-Domain Analysis

- 2<sup>nd</sup>-Order System – Step Response Analysis

- For unity-gain  $K = 1$  and  $x_i(t) = B u(t)$ :

$$PO = \frac{\text{Peak value} - \text{Final value}}{\text{Final value}} \times 100\% = 100 e^{-\pi\zeta/\sqrt{1-\zeta^2}}$$

$$a_1 = B e^{-\zeta\omega_n t_p}, \quad t_p = \frac{\pi}{\omega_d} = \frac{\pi}{\omega_n \sqrt{1-\zeta^2}}$$

$$PO = \frac{a_1}{B} \times 100, \quad t_p = \text{Time of first peak},$$

- Clearly  $\zeta$  can be expressed i.t.o.  $PO$ .



# Time-Domain Analysis

- 2<sup>nd</sup>-Order System – Step Performance Spec.
  - Rise time:  $t_r = t|_{x_o=0.9B} - t|_{x_o=0.1B}$
  - 2% Settling time:  $t_s = \frac{\ln 50}{\zeta \omega_n}$
  - Delay time:  $t_d = t|_{x_o=0.5B}$
  - Contours in the s-plane (Kuo, *Automatic Control Systems*, 7<sup>th</sup> ed., Fig 7.15, p. 391 & Lathi, *Signals, Systems and Controls*, pp. 253–255.)

# Tutorial Exercises & Homework

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- Tutorial Exercises

- Burns, Examples 3.9 and 3.10

- Web Surfing – Exploring 2<sup>nd</sup>-Order Systems

<http://www.facstaff.bucknell.edu/mastascu/econtrolhtml/SysDyn/SysDyn2.html>

- Homework

- Burns: Example 3.7, p 51, Sec 3.4, 3.5, 3.6.4, 3.8

# Conclusion


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- Briefly Reviewed Time Domain Analysis
- Focused on the Prototype 2<sup>nd</sup>-order system
- Burns, Sec 3.4, 3.5, 3.6.4, 3.8 (Self-study!)
- Tutorial Exercises & Homework

**Next Attraction! – Miss It & You'll Miss Out!**

- Closed-Loop Control Systems  
(Burns, Chapter 4)

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**Thank you!**  
**Any Questions?**