CONTROL I

ELEN3016

Digital Controller Design

(Lecture 24)

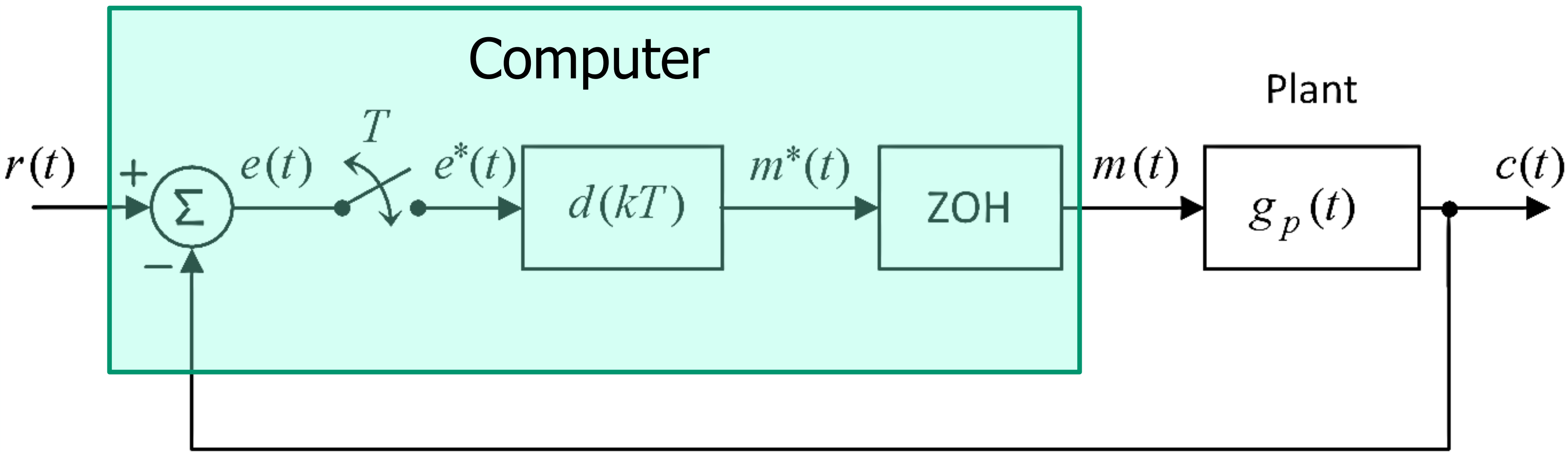
Overview

- First Things First!
- Digital Controller Design

  (Classical Methods, Numerical Methods, Analytical Design, Optimum Response Digital Design)
- Tutorial Exercises & Homework

Digital Controller Design

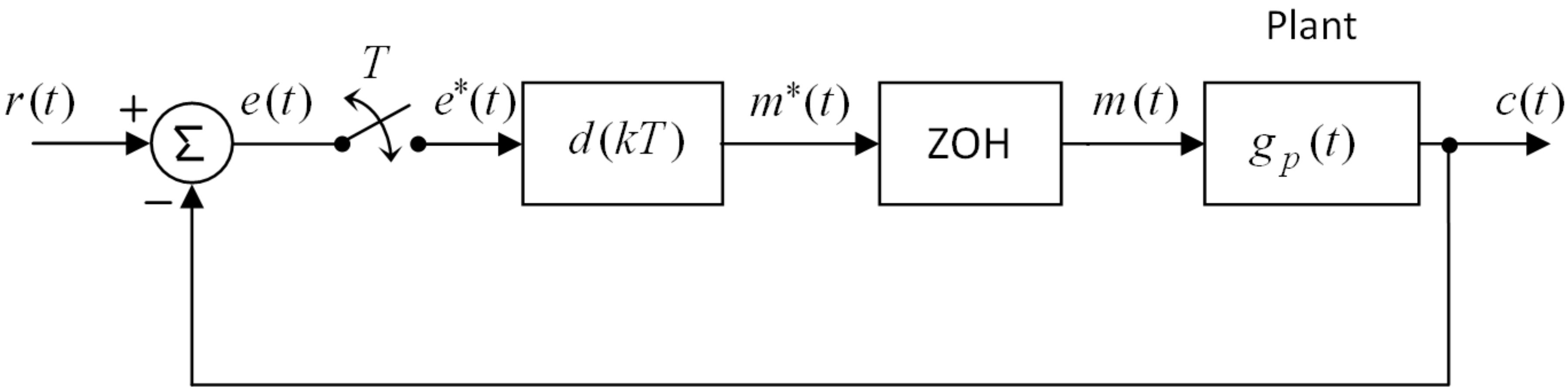
- Computer-Controlled System



Note: More correctly (but equivalently) the sampler should be moved backwards to the reference input and feedback input.

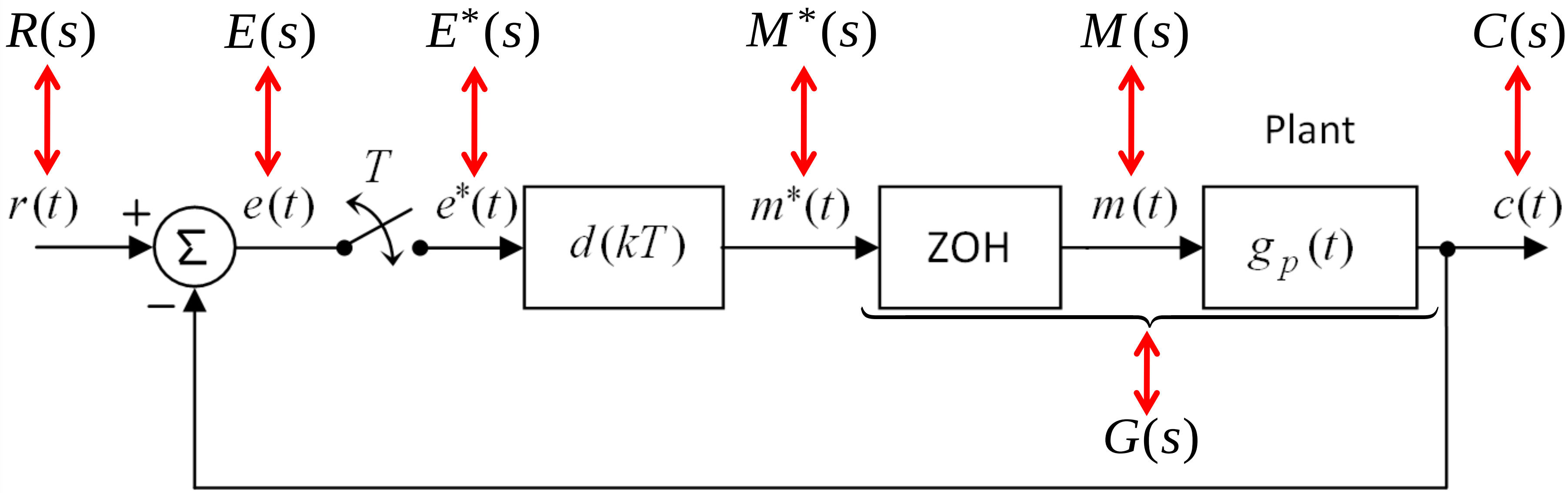
Digital Controller Design

- Computer-Controlled System



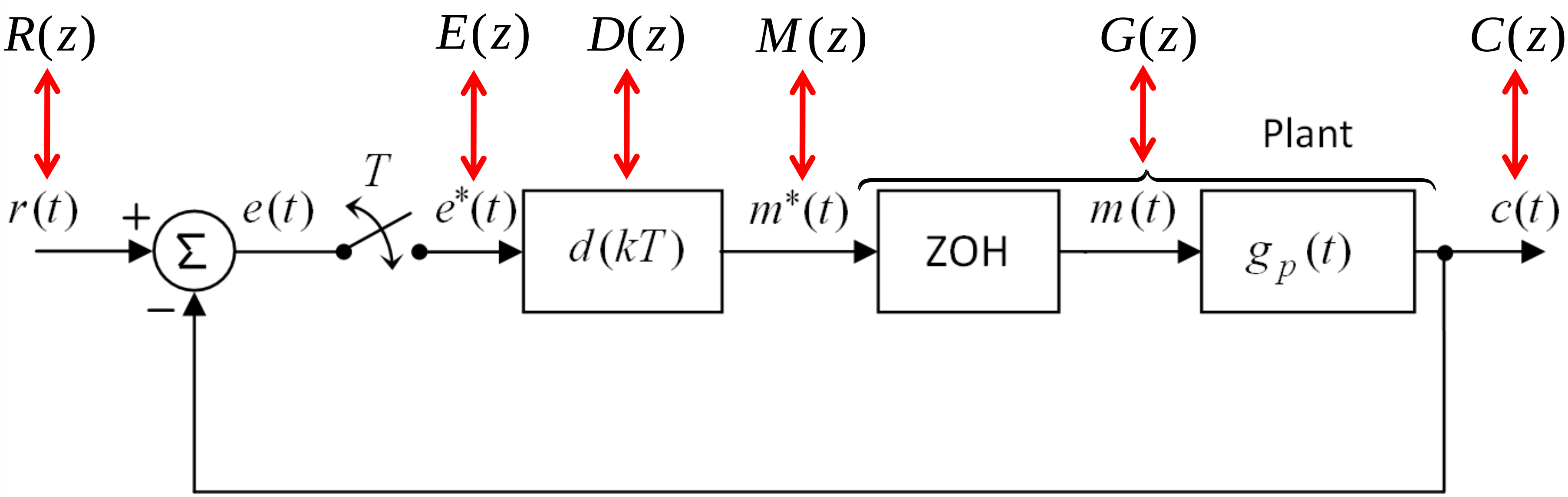
Digital Controller Design

- Computer-Controlled System



Digital Controller Design

- Computer-Controlled System



Digital Controller Design

- Computer-Controlled System cont'd
  - During the k th sampling interval

$$m(t) = m(kT), \qquad kT \le t < (k+1)T$$

  - By letting $G(s)$ represent the ZOH & plant combination, $C(s) = G(s)M^*(s)$ and starring yields

$$C^*(s) = G^*(s)M^*(s)$$

and

$$C(z) = G(z)M(z)$$

Digital Controller Design

- Computer-Controlled System cont'd
  - Z transform of the digital controller is

$$M(z) = D(z)\,E(z)$$

  - Z transform of the output is

$$C(z) = G(z)\overbrace{D(z)E(z)}^{M(z)} = G(z)D(z)\overbrace{\left(R(z) - C(z)\right)}^{E(z)}$$

giving

$$C(z) = \frac{D(z)G(z)}{1 + D(z)G(z)}R(z)$$

Digital Controller Design

- Digital Controller Design via Classical Methods
  - Transform $G(z)$ to an equivalent s-plane called the w-plane using the bilinear transformation $z = (w+1)/(w-1)$.
  - Apply any classical design procedure.
  - Transform back to the z-plane using the inverse transformation $w = (z+1)/(z-1)$.

Digital Controller via Classical Design

- Example (Phillips & Nagle)
  - Plant, $G_p(s) = \dfrac{8}{s(s+1)(\frac{1}{2}s+1)}$.
  - Specifications:

    Unit DC gain.

    Phase margin: $PM = 55^\circ$

Digital Controller via Classical Design

- Example cont'd
  - Selection of T:

    System's fastest time constant is 0.5 s. A rule of thumb is to choose T one-tenth of the fastest time constant i.e. $T = 0.05$ s.

  - ZOH–Plant combination

$$G(s) = \underbrace{\left(1 - e^{-sT}\right)}_{G_1(s)} \frac{1}{\underbrace{s^2(s+1)(0.5s+1)}_{G_2(s)}}$$

Digital Controller via Classical Design

- Example cont'd
  - Discrete part of ZOH

$$G_1(s) = 1 - e^{-0.05s} \qquad (T = 0.05\text{s})$$

$$G_1(z) = \mathcal{Z}\left[1 - e^{-0.05s}\right] = 1 - z^{-1} = \frac{z-1}{z}$$

  - Continuous part of ZOH and plant

$$G_2(s) = \frac{1}{s^2(s+1)(0.5s+1)} = \frac{1}{s^2} - \frac{1.5}{s} + \frac{2}{s+1} - \frac{0.5}{s+2}$$

Digital Controller via Classical Design

- Example cont'd
  - Taking the Z-transform then yields

$$G_2(z) = \mathscr{Z}\left[\frac{1}{s^2} - \frac{1.5}{s} + \frac{2}{s+1} - \frac{0.5}{s+2}\right]$$

$$= \frac{0.005z}{(z-1)^2} - \frac{1.5z}{z-1} + \frac{2z}{z-0.9512} - \frac{0.5z}{z-0.9048}$$

Digital Controller via Classical Design

- Example cont'd
  - ZOH & plant combination pulse transfer function becomes

$$G(z) = G_1(z)G_2(z) =$$

$$= \frac{z-1}{z}\left[\frac{0.005z}{(z-1)^2} - \frac{1.5z}{z-1} + \frac{2z}{z-0.9512} - \frac{0.5z}{z-0.9048}\right]$$

$$= \frac{z^3 - 2.9962z^2 + 2.9862z - 0.9899}{z^3 - 2.8560z^2 + 2.7166z - 0.8606}$$

Digital Controller via Classical Design

- Example cont'd
  - ZOH & plant combination pulse transfer function becomes

$$G(w) = G(z)\Big|_{z=\frac{w+1}{w-1}}$$

$$\vdots$$

$$= \frac{w^3 - 541.01w^2 + 1733.38w + 343206.69}{w^3 - 141.01w^2 + 25730.18w + 663209.94}$$

... to be completed by you! ☺

Digital Controller Design

- Digital Controller Design via Numerical Methods
  - Transform ordinary differential equations to equivalent difference equations using various approximations for integrals and derivatives studied in numerical methods.
  - Particularly useful for deriving discrete equivalent PID controllers from the standard continuous time PID controller.

Digital Controller Design

- Analytical Design
  - Expresses the controller pulse transfer function $D(z)$ in terms of the ZOH & plant pulse transfer function $G(z)$ and desired closed-loop pulse transfer function $C(z)/R(z)$.

Digital Controller Design

- Analytical Design cont'd
  - Recall from an earlier slide that

$$C(z)=G(z)D(z)\overbrace{\left(R(z)-C(z)\right)}^{E(z)}$$

  - From this the expression for $D(z)$ is

$$D(z)=\frac{C(z)}{G(z)\left(R(z)-C(z)\right)}=\frac{1}{G(z)}\frac{C(z)/R(z)}{\left(1-C(z)/R(z)\right)}$$

Digital Controller Design

- Analytical Design cont'd
  - Recall from an earlier slide that

$$C(z) = G(z)D(z)\overbrace{\left(R(z) - C(z)\right)}^{E(z)}$$

  - From this the expression for $D(z)$ is

$$D(z) = \frac{C(z)}{G(z)\left(R(z) - C(z)\right)} = \frac{1}{G(z)} \frac{C(z)/R(z)}{\left(1 - C(z)/R(z)\right)}$$

Analytical Design

- Example (Raven)
  - Plant, $G_p(s) = \dfrac{(s+2)}{s(s+1)}$.
  - Specifications:

    Sampling period: $T = 1.0$ s

    Desired response required: $c(t) = 5(1 - e^{-2t})$

Analytical Design

- Example cont'd
  - ZOH–Plant combination

$$G(s)=\underbrace{\left(1-e^{-sT}\right)}_{G_1(s)}\underbrace{\frac{(s+2)}{s^2(s+1)}}_{G_2(s)} \qquad (T=1)$$

  - Discrete part of ZOH

$$G_1(s)=1-e^{-s}$$

$$G_1(z)=\mathcal{Z}\left[1-e^{-s}\right]=1-z^{-1}=\frac{z-1}{z}$$

Analytical Design

- Example cont'd
  - Plant and continuous part of ZOH

$$G_2(s) = \frac{(s+2)}{s^2(s+1)} = \frac{2}{s^2} - \frac{1}{s} + \frac{1}{s+1}$$

$$G_2(z) = \mathcal{Z}\left[\frac{2}{s^2} - \frac{1}{s} + \frac{1}{s+1}\right] = \frac{2z}{(z-1)^2} - \frac{z}{z-1} + \frac{z}{z-0.368}$$

  - ZOH & plant combination pulse transfer function

$$G(z) = G_1(z)G_2(z) = \frac{1.368z - 0.104}{z^2 - 1.368z + 0.368}$$

Analytical Design

- Example cont'd
  - The Laplace transform of the desired response is

$$C(s) = 5\left(\frac{1}{s} - \frac{1}{s+2}\right)$$

and

$$C(z) = 5\mathcal{Z}\left[\frac{1}{s} - \frac{1}{s+2}\right] = 5\left[\frac{z}{z-1} + \frac{z}{z-0.135}\right] = \frac{4.32z}{(z-1)(z-0.135)}$$

  - From this we derive the ratios

$$\frac{C(z)}{R(z)} = \frac{4.32}{z-0.135} \quad \text{and} \quad 1 - \frac{C(z)}{R(z)} = \frac{z-4.45}{z-0.135}$$

Analytical Design

- Example cont'd
  - Substituting the above expressions into the formula for $D(z)$ yields

$$D(z) = \frac{z^2 - 1.368z + 0.368}{1.368z - 0.104} \frac{4.32}{z - 0.135}$$

$$= \frac{4.32z^2 - 5.91z + 1.59}{1.368z^2 - 6.202z + 0.46}$$

$$= \frac{3.16 - 4.32z^{-1} + 1.16z^{-2}}{1 - 4.53z^{-1} + 0.34z^{-2}} = \frac{M(z)}{E(z)}$$

Analytical Design

- Example cont'd
  - Cross multiplication gives

$$\left(1-4.53z^{-1}+0.34z^{-2}\right)M(z)=\left(3.16-4.32z^{-1}+1.16z^{-2}\right)E(z)$$

  - The time domain equation for the controller is

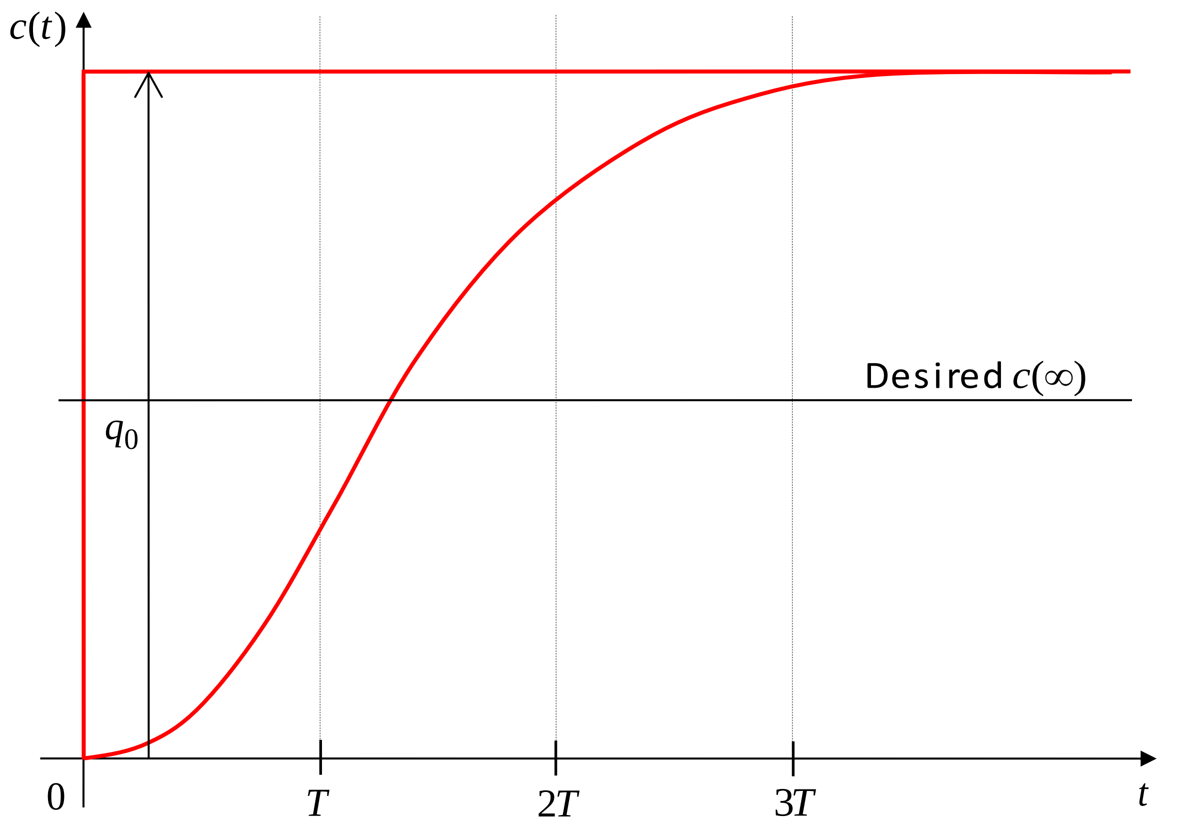
$$m(k)=3.16\,e(k)-4.32\,e(k-1)+1.16\,e(k-2)+$$
$$+4.53\,m(k-1)-0.34\,m(k-2)$$

Digital Controller Design

- Optimum Response Design
  - *Optimum*: The closed-loop system responds to a step input in the minimum time with no overshoot and no steady-state error.
  - For a system of order N the step response settles to the desired final value after $N+1$ sample instants.
  - Proposed by R.E. Kalman in the 1954.

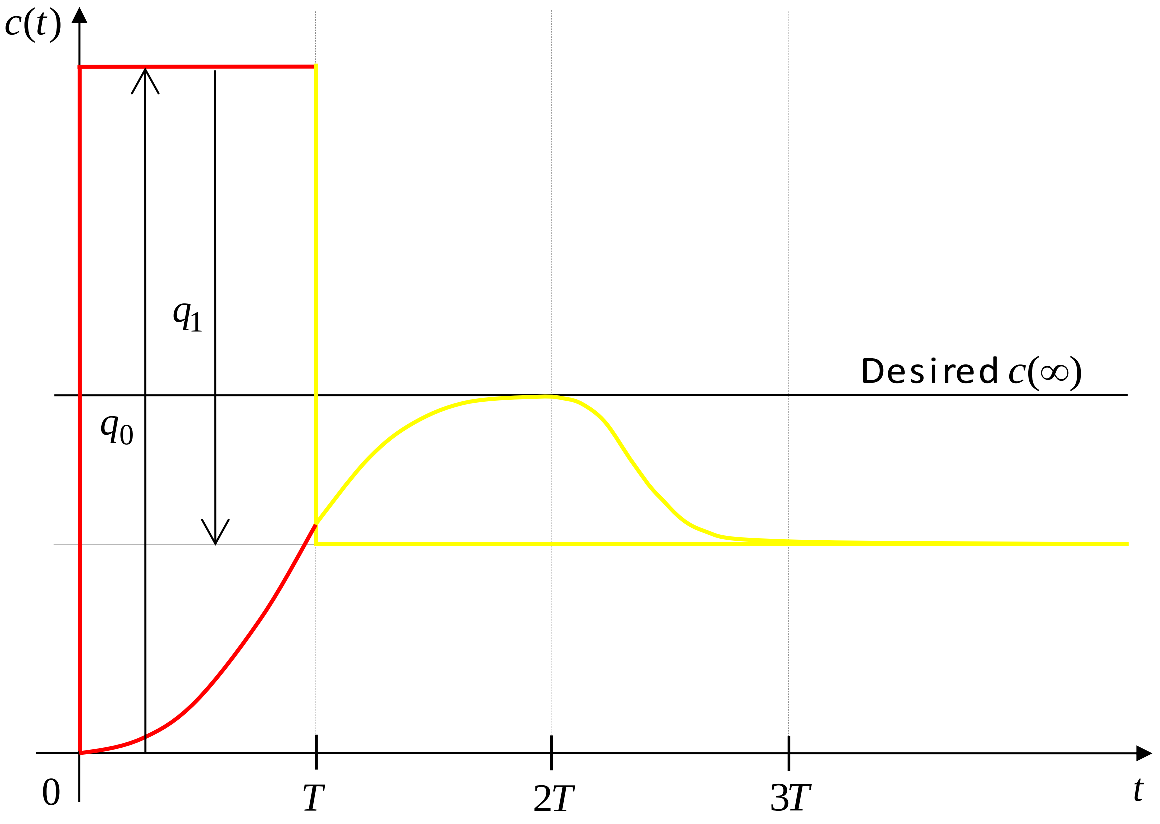
2nd Order Optimum Response

First sampling period excitation



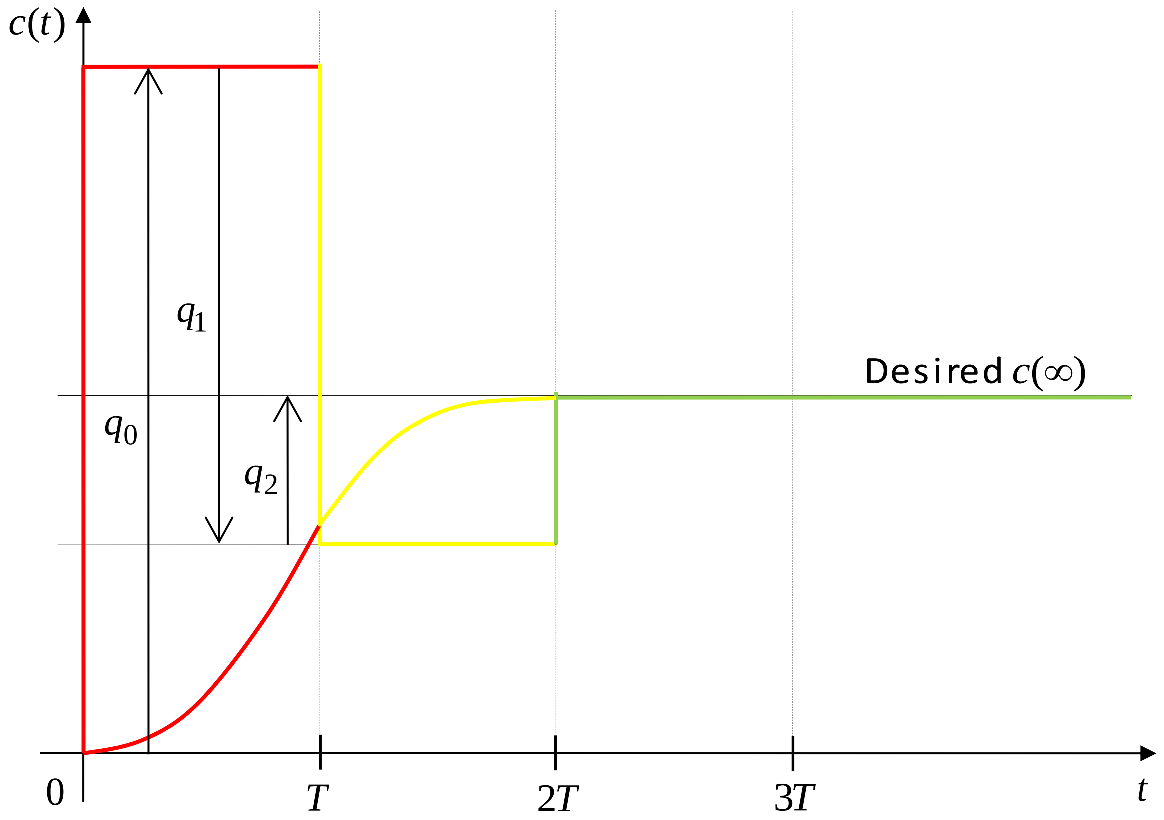
2^{nd} Order Optimum Response

Second sampling period excitation



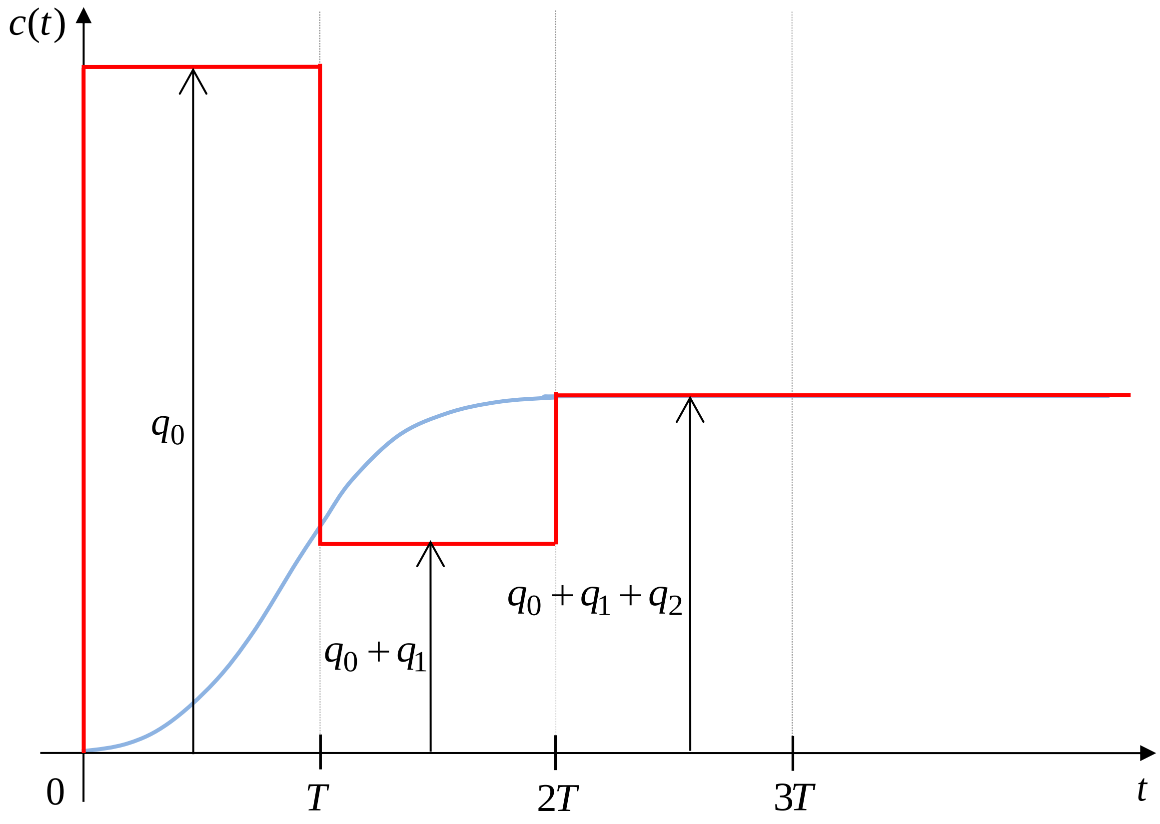
2^{nd} Order Optimum Response

Third sampling period excitation



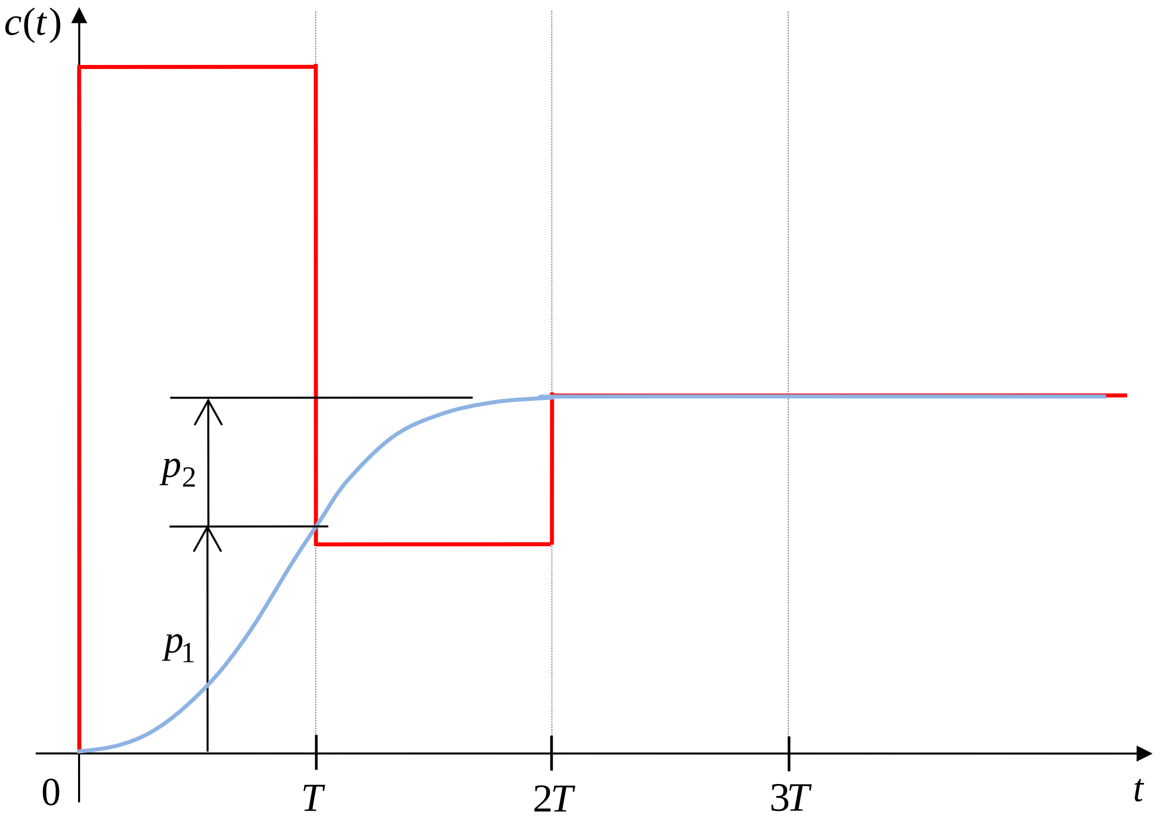
2nd Order Optimum Response

Complete plant excitation & output response



2^{nd} Order Optimum Response

Complete plant excitation & output response



2^{nd} Order Optimum Response

- Derivation (Raven)
  - For unit step input

$$R(z) = \frac{1}{1-z^{-1}} = 1 + z^{-1} + z^{-2} + z^{-3} + \ldots$$

  - The ZOH's output is (see earlier graphical depiction)

$$M(z) = q_0 + (q_0 + q_1)z^{-1} + (q_0 + q_1 + q_2)(z^{-2} + z^{-3} + \ldots)$$

  - Long division yields (see next slide)

$$\frac{M(z)}{R(z)} = \ldots = q_0 + q_1 z^{-1} + q_2 z^{-2}$$

2^{nd} Order Optimum Response

- Derivation (Raven)
  - Long division:

$$1+z^{-1}+z^{-2}+z^{-3}+\ldots \overline{\big)\, q_0+(q_0+q_1)z^{-1}+(q_0+q_1+q_2)(z^{-2}+z^{-3}+\ldots)}$$

2nd Order Optimum Response

- Derivation (Raven)
  - Long division:

$$
\begin{array}{r|l}
 & q_0 \\
\cline{2-2}
1+z^{-1}+z^{-2}+z^{-3}+\ldots & q_0+(q_0+q_1)z^{-1}+(q_0+q_1+q_2)(z^{-2}+z^{-3}+\ldots) \\
 & q_0 + \quad q_0 z^{-1} + \quad q_0(z^{-2}+z^{-3}+\ldots) \\
\cline{2-2}
 & 0 + \quad q_1 z^{-1} + \quad (q_1+q_2)(z^{-2}+z^{-3}+\ldots)
\end{array}
$$

2^{nd} Order Optimum Response

- Derivation (Raven)
  - Long division:

$$
\begin{array}{r|l}
 & q_0 + q_1 z^{-1} \\
\hline
1 + z^{-1} + z^{-2} + z^{-3} + \ldots & q_0 + (q_0 + q_1) z^{-1} + (q_0 + q_1 + q_2)(z^{-2} + z^{-3} + \ldots) \\
 & q_0 + \quad q_0 z^{-1} + \quad q_0 (z^{-2} + z^{-3} + \ldots) \\
\cline{2-2}
 & 0 + \quad q_1 z^{-1} + \quad (q_1 + q_2)(z^{-2} + z^{-3} + \ldots) \\
 & \quad q_1 z^{-1} + \quad q_1 (z^{-2} + z^{-3} + \ldots) \\
\cline{2-2}
 & \quad 0 + \quad q_2 (z^{-2} + z^{-3} + \ldots)
\end{array}
$$

2nd Order Optimum Response

- Derivation (Raven)
  - Long division:

$$
\begin{array}{r|l}
 & q_0 + q_1 z^{-1} + q_2 z^{-2} \\
\hline
1 + z^{-1} + z^{-2} + z^{-3} + \ldots & q_0 + (q_0 + q_1) z^{-1} + (q_0 + q_1 + q_2)(z^{-2} + z^{-3} + \ldots) \\
 & q_0 + \quad q_0 z^{-1} + \quad q_0 (z^{-2} + z^{-3} + \ldots) \\
\cline{2-2}
 & 0 + \quad q_1 z^{-1} + \quad (q_1 + q_2)(z^{-2} + z^{-3} + \ldots) \\
 & \quad q_1 z^{-1} + \quad q_1 (z^{-2} + z^{-3} + \ldots) \\
\cline{2-2}
 & \quad 0 + \quad q_2 (z^{-2} + z^{-3} + \ldots) \\
 & \quad q_2 (z^{-2} + z^{-3} + \ldots) \\
\cline{2-2}
 & \quad 0
\end{array}
$$

2^{nd} Order Optimum Response

- Derivation (Raven)
  - Long division:

$$
\begin{array}{r|lclcl}
 & q_0 + q_1 z^{-1} + q_2 z^{-2} & & & & \\ \cline{2-6}
1 + z^{-1} + z^{-2} + z^{-3} + \ldots & q_0 & + & (q_0 + q_1) z^{-1} & + & (q_0 + q_1 + q_2)(z^{-2} + z^{-3} + \ldots) \\
 & q_0 & + & q_0 z^{-1} & + & q_0 (z^{-2} + z^{-3} + \ldots) \\ \cline{2-6}
 & 0 & + & q_1 z^{-1} & + & (q_1 + q_2)(z^{-2} + z^{-3} + \ldots) \\
 & & & q_1 z^{-1} & + & q_1 (z^{-2} + z^{-3} + \ldots) \\ \cline{4-6}
 & & & 0 & + & q_2 (z^{-2} + z^{-3} + \ldots) \\
 & & & & & q_2 (z^{-2} + z^{-3} + \ldots) \\ \cline{6-6}
 & & & & & 0
\end{array}
$$

$$
\frac{M(z)}{R(z)} = \ldots = q_0 + q_1 z^{-1} + q_2 z^{-2}
$$

2^{nd} Order Optimum Response

- Derivation cont'd
  - The plant's sampled output is (see earlier graphical depiction)

$$C(z) = p_1 z^{-1} + (p_1 + p_2)(z^{-2} + z^{-3} + z^{-4} + \ldots)$$

  - Long division yields

$$\frac{C(z)}{R(z)} = \ldots = p_1 z^{-1} + p_2 z^{-2}$$

  - Pulse transfer function of the ZOH and plant combination is

$$G(z) = \frac{C(z)}{M(z)} = \frac{C(z)/R(z)}{M(z)/R(z)} = \frac{p_1 z^{-1} + p_2 z^{-2}}{q_0 + q_1 z^{-1} + q_2 z^{-2}}$$

2^{nd} Order Optimum Response

- Derivation cont'd
  - The plant's output is

$$C(z) = p_1 z^{-1} + (p_1 + p_2)(z^{-2} + z^{-3} + z^{-4} + \ldots)$$

  - Pulse transfer function of the digital controller is

$$D(z) = \frac{M(z)}{E(z)} = \frac{M(z)}{R(z) - C(z)} = \frac{M(z)/R(z)}{1 - C(z)/R(z)} = \frac{q_0 + q_1 z^{-1} + q_2 z^{-2}}{1 - p_1 z^{-1} - p_2 z^{-2}}$$

  - The inverse Z-transform gives

$$m(k) = q_0\, e(k) + q_1\, e(k-1) + q_2\, e(k-2) + \\ + p_1 m(k-1) + p_2\, m(k-2)$$

2nd Order Optimum Response

- Derivation cont'd
  - The FVT yields the steady-state of the plant output as

$$c(\infty) = \lim_{z \to 1} \frac{z-1}{z} C(z) = \ldots = p_1 + p_2$$

  - The steady-state of $m(t)$ is

$$m(\infty) = \lim_{z \to 1} \frac{z-1}{z} M(z) = \ldots = q_0 + q_1 + q_2$$

  - The closed-loop system steady-state gain is

$$K = \frac{c(\infty)}{r(\infty)} = p_1 + p_2$$

2^{nd} Order Optimum Response

- Derivation cont'd
  - The plant steady-state gain is

$$K_p = \frac{c(\infty)}{m(\infty)} = \frac{p_1 + p_2}{q_0 + q_1 + q_2}$$

  - For a 2^{nd} order plant the closed-loop pulse transfer function has the form

$$G(z) = \frac{a_1 z^{-1} + a_2 z^{-2}}{b_0 + b_1 z^{-1} + b_2 z^{-2}}$$

  - From this we obtain

$$K = p_1 + p_2 = k(a_1 + a_2) \qquad k = \frac{K}{a_1 + a_2}$$

2^{nd} Order Optimum Response

- Interpretation
  - Considering the product of $D(z)$ and $G(z)$ namely

$$D(z)G(z) = \frac{q_0 + q_1 z^{-1} + q_2 z^{-2}}{1 - p_1 z^{-1} - p_2 z^{-2}} \frac{p_1 z^{-1} + p_2 z^{-2}}{q_0 + q_1 z^{-1} + q_2 z^{-2}}$$

$$= \frac{p_1 z^{-1} + p_2 z^{-2}}{1 - p_1 z^{-1} - p_2 z^{-2}}$$

shows that the controller zeros cancel the ZOH & plant poles and that both poles and zeros of the compensated open-loop system $D(z)G(z)$ depend on p_1 and p_2 exclusively.

2^{nd} Order Optimum Response

- Interpretation cont'd
  - The closed-loop compensated system now has as pulse transfer function

$$\frac{C(z)}{R(z)} = \frac{D(z)G(z)}{1+D(z)G(z)} = \ldots = p_1 z^{-1} + p_2 z^{-2}$$

    and clearly is of finite impulse response (FIR) type even though the compensated open-loop system is of infinite impulse response (IIR) type.
  - This FIR structure explains why the closed-loop compensated system's step response settles in a finite number of samples.

2^{nd} Order Optimum Response

- Interpretation cont'd
  - When the sampling period T is sufficiently shorter than the fastest time constant of the closed-loop compensated system the step response of the closed-loop compensated system is guaranteed to be settled for $t \geq 2T$.

2^{nd} Order Optimum Response

- Example (Raven)
  - Plant, $G_p(s) = \dfrac{(s+2)}{s(s+1)}$.
  - Specifications:

    Sampling period: $T = 1.0$ s

    Optimum response required.

    Closed-loop steady-state gain: $K = 5$

2^{nd} Order Optimum Response

- Example cont'd
  - ZOH–Plant combination

$$G(s) = \underbrace{\left(1 - e^{-sT}\right)}_{G_1(s)} \underbrace{\frac{(s+2)}{s^2(s+1)}}_{G_2(s)}$$

  - Discrete part of ZOH

$$G_1(s) = 1 - e^{-s} \qquad\qquad (T = 1)$$

$$G_1(z) = \mathfrak{Z}\left[1 - e^{-s}\right] = 1 - z^{-1} = \frac{z-1}{z}$$

2^{nd} Order Optimum Response

- Example cont'd
  - Plant and continuous part of ZOH

$$G_2(s)=\frac{(s+2)}{s^2(s+1)}=\frac{2}{s^2}-\frac{1}{s}+\frac{1}{s+1}$$

$$G_2(z)=\mathcal{Z}\left[\frac{2}{s^2}-\frac{1}{s}+\frac{1}{s+1}\right]=\frac{2z}{(z-1)^2}-\frac{z}{z-1}+\frac{z}{z-0.368}$$

  - ZOH & plant combination pulse transfer function

$$G(z)=G_1(z)G_2(z)=\frac{1.368z-0.104}{z^2-1.368z+0.368}$$

2nd Order Optimum Response

- Example cont'd
  - From the numerator

$$a_1 + a_2 = 1.368 - 0.104 = 1.264$$

  - Desired gain

$$K = p_1 + p_2 = 5 = (a_1 + a_2)k = 1.264k$$

$$k = \frac{5}{1.264} = 3.96$$

  - For the required gain the plant becomes

$$G(s) = \frac{k\left(1.368z - 0.104\right)}{k\left(z^2 - 1.368z + 0.368\right)} = \frac{5.42 - 0.42z^{-1}}{3.96 - 5.42z^{-1} + 1.46z^{-2}}$$

2nd Order Optimum Response

- Example cont'd
  - Equating

$$\frac{p_1 z^{-1} + p_2 z^{-2}}{q_0 + q_1 z^{-1} + q_2 z^{-2}} = \frac{5.42 - 0.42 z^{-1}}{3.96 - 5.42 z^{-1} + 1.46 z^{-2}}$$

and comparing coefficients yields

$$p_1 = 5.42,\ \ p_2 = -0.42,\ \ q_0 = 3.96,\ \ q_1 = -5.42,\ \ q_2 = 1.46\,.$$

  - Verification:

$$p_1 + p_2 = 5.42 - 0.42 = 5.$$

2nd Order Optimum Response

- Example cont'd
  - The time domain equation for the controller is

$$m(k) = 3.96\,e(k) - 5.42\,e(k-1) + 1.46\,e(k-2) + \\ + 5.42\,m(k-1) - 0.42\,m(k-2)$$

Tutorial Exercises & Homework

- Tutorial Exercises
  - To be announced at the beginning of the tut session.
- Homework
  - Study all relevant sections in Burns.

Conclusion

- Digital Controllers – Overview
- Analytical Design – Theory & Example
- Optimum Response Design – Theory & Example
- For general optimum response design refer to F.H. Raven "Automatic Control Engineering" 5th edition, pp 494–496 (Optional)
- Tutorial Exercises & Homework

Conclusion

- … a last few questions …
  - Was the course a mathematical as you have been told?

Conclusion

- … a last few questions …
  - Was the course a mathematical as you have been told?
  - Do you think Control has application other than in just Control Engineering? (Should all E&I engineers be taught Control?)

Conclusion

- … a last few questions …
  - Was the course a mathematical as you have been told?
  - Do you think Control has application other than in just Control Engineering? (Should all E&I engineers be taught Control?)
  - What would you have like to see in the course?

**Thank you
for your interest!**