CONTROL I

ELEN3016

Discrete-Time Systems Properties

(Lecture 23)

Overview

- First Things First!
- Transient response
- Stability
- Root locus
- Discrete-time to continuous-time conversion
  - Filters and Zero-Order Hold (ZOH)
- Tutorial Exercises & Homework

First Things First!

- None from my side.
- From your side?

Discrete Systems Properties

- Transient Response

Relationship between paths in the s-plane and the z-plane follows from

$$z = e^{sT} = e^{j(\sigma + j\omega)T} = e^{j\sigma T} e^{j\omega T}.$$

... consider different contours in the s-plane and their dual in the z-plane.

Discrete Systems Properties

- Transient Response cont'd
  - For $z = e^{sT}$ with $T > 0$ we have that

$$|z| = \left|e^{(\sigma + j\omega)T}\right| = \left|e^{\sigma T}\right| \underbrace{\left|e^{j\omega T}\right|}_{=1} = e^{\sigma T}$$

  - For $\sigma > 0$ we have that $e^{\sigma T} > 1.$
  - For $\sigma = 0$ we have that $e^{\sigma T} = 1.$
  - For $\sigma < 0$ we have that $0 < e^{\sigma T} < 1.$

Discrete Systems Properties

- Transient Response cont'd

  We conclude that

  - the LHP of the s-plane maps to the interior of the unit circle in the z-plane;
  - the $j\omega$-axis in the s-plane maps to the circumference of the unit circle in the z-plane;
  - the RHP of the s-plane maps to the exterior of the unit circle in the z-plane.

Discrete Systems Properties

- Stability

Consider the characteristic equation

$$a_n z^n + a_{n-1} z^{n-1} + \ldots + a_1 z + a_0 = 0$$

Two methods exist for stability analysis:

– Jury test (consult Burns),

– Routh-Hurwitz test.

Discrete Systems Properties

- Stability – Routh-Hurwitz Test
  - The substitution $z = \frac{w+1}{w-1}$ (bilinear transformation maps the interior (exterior resp.) of the unit circle onto the LHP (RHP resp.) of the w-plane. The Char. Eq. becomes

$$a_n\left(\frac{w+1}{w-1}\right)^n + a_n\left(\frac{w+1}{w-1}\right)^{n-1} + \ldots + a_1\left(\frac{w+1}{w-1}\right) + a_0 = 0$$

$$\Rightarrow a_n(w+1)^n + a_n(w+1)^{n-1}(w-1) + \ldots$$
$$+ a_1(w+1)(w-1)^{n-1} + a_0(w-1)^n = 0$$

Discrete Systems Properties

- Stability – Routh-Hurwitz Test cont'd
  - The final result has the form

$$c_n w^n + c_{n-1} w^{n-1} + \ldots + c_1 w + c_0 = 0$$

  - Now apply the Routh-Hurwitz test to this polynomial.

Discrete Systems Properties

- Example – Routh-Hurwitz Test
  - Consider the closed-loop characteristic equation

$$1+GH(z)=1+\frac{0.092K(z-0.7174)}{(z-1)(z-0.368)}=0$$

  - This may be expressed as

$$z^2-1.368z+0.368+0.092Kz+0.7174\cdot 0.092K=0$$

$$z^2+(0.092K-1.368)z+(0.66K+0.368)=0$$

Discrete Systems Properties

- Example – Routh-Hurwitz Test cont'd
  - Next transform to the w-plane using

$$z = \frac{w+1}{w-1}$$

  - This yields

$$0.158Kw^2 + 2(0.632 - 0.066K)w + \\ + (2.736 - 0.026K) = 0$$

Discrete Systems Properties

- Example – Routh-Hurwitz Test cont'd
  - Routh Array

w^0	$2.736-0.026K$	0
w^1	$2(0.632-0.066K)$	0
w^2	$0.158K$	$2.736-0.026K$

Discrete Systems Properties

- Example – Routh-Hurwitz Test cont'd
  - To have no sign changes in the first column requires

$$0.158K > 0 \quad \Rightarrow \quad K > 0$$

$$0.632 - 0.066K \quad \Rightarrow \quad K < 9.576$$

$$2.736 - 0.026K \quad \Rightarrow \quad K < 105.23$$

  - System stable for

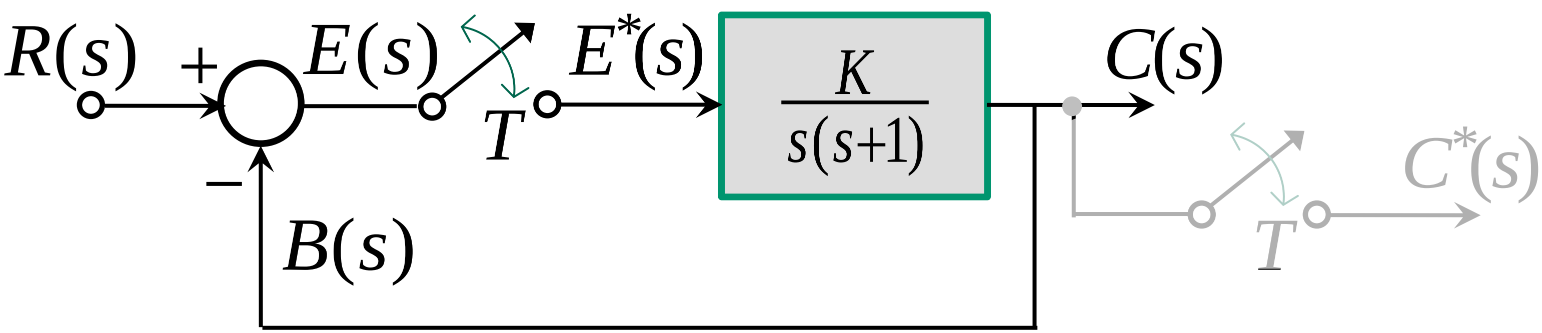
$$0 < K < \min(9.576, 105.23) = 9.576$$

Discrete Systems Properties

- Root Locus
  - The root locus rules, as for continuous systems, applies to the discrete Char. Eq.
  - The only difference is the interpretation of segments of loci that correspond to stable response.

Discrete Systems Properties

- Root Locus – Example



  - The closed-loop pulse transfer function is

$$\frac{C(z)}{R(z)} = \frac{G(z)}{1+G(z)} \quad \text{with} \quad G(s) = \frac{K}{s(s+1)}\,.$$

Discrete Systems Properties

- Root Locus – Example cont'd

We proceed to calculate $G(z)$.

Using PFE, namely

$$G(s) = \frac{K}{s(s+1)} = K\left(\frac{1}{s} - \frac{1}{s+1}\right)$$

we obtain

$$G(z) = \mathfrak{Z}\left[G(s)\right] = K\mathfrak{Z}\left(\frac{1}{s} - \frac{1}{s+1}\right) = K\left(\mathfrak{Z}\left(\frac{1}{s}\right) - \mathfrak{Z}\left(\frac{1}{s+1}\right)\right) =$$

$$= K\left(\frac{z}{z-1} - \frac{z}{z-e^{-T}}\right) = K\frac{z\left(1-e^{T}\right)}{(z-1)\left(z-e^{-T}\right)}$$

Discrete Systems Properties

- Root Locus – Example cont'd

Char. Eq. of the closed-loop system is

$$1 + G(z) = 0$$

$$(z-1)\left(z-e^{-T}\right) + K\left(1-e^{-T}\right)z = 0$$

Open-loop zeros: $z = 0$

Open-loop poles: $z = 1, \quad z = e^{-T}$

Real axis segments implies break-in/away pts.

Discrete Systems Properties

- Root Locus – Example cont'd

Break-in & breakaway points:

$$-K\left(1-e^{-T}\right)=\frac{(z-1)\left(z-e^{-T}\right)}{z}$$

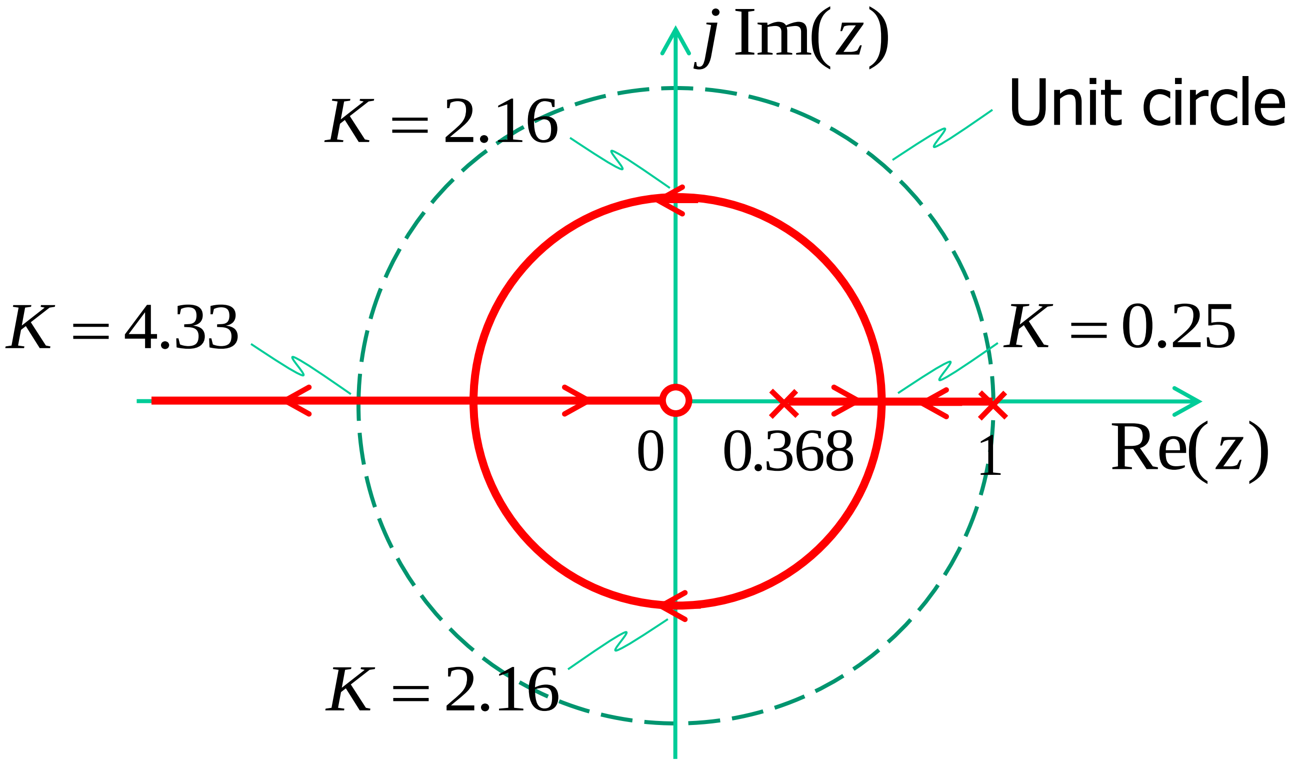
$$-\frac{d}{dz}\left(\left(1-e^{-T}\right)K\right)=\frac{d}{dz}\left(\frac{(z-1)\left(z-e^{-T}\right)}{z}\right)=\frac{z^2-e^{-T}}{z^2}=0$$

$$\Rightarrow \quad z=\pm e^{-\frac{T}{2}}$$

Discrete Systems Properties

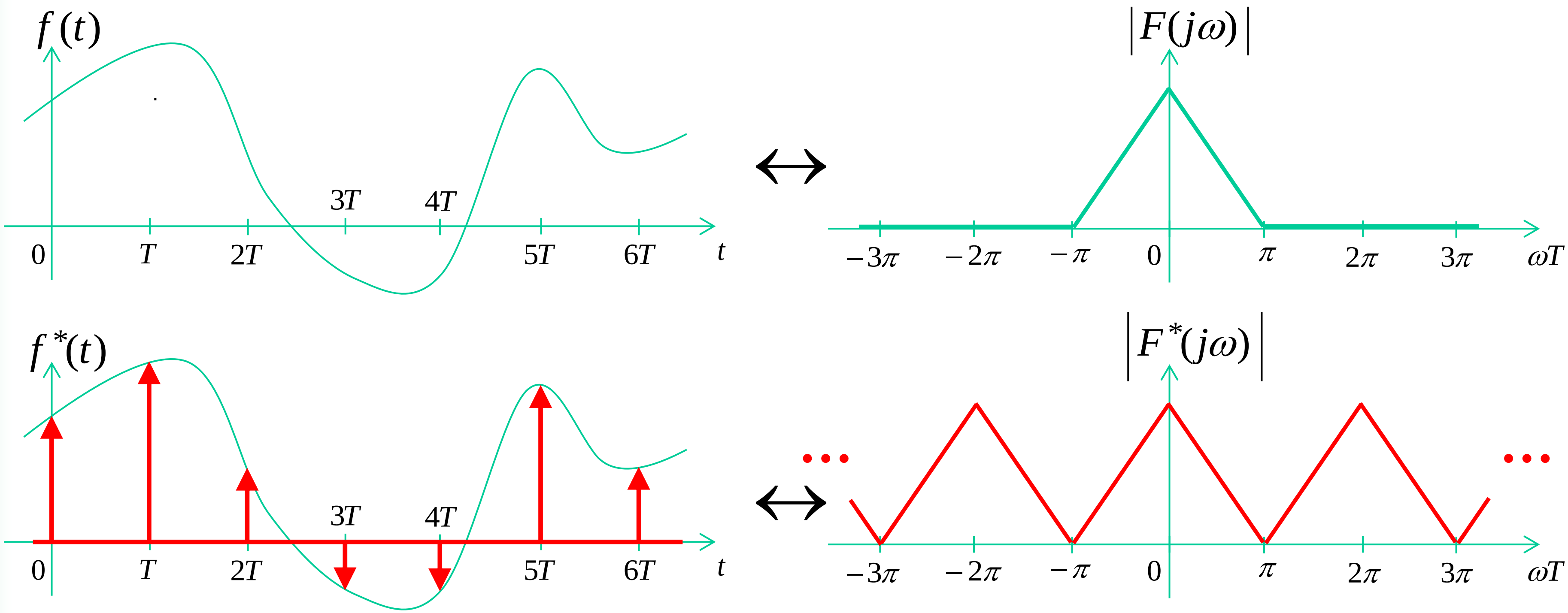
- Root Locus – Example cont'd

For $T = 1$ the root locus is shown below.



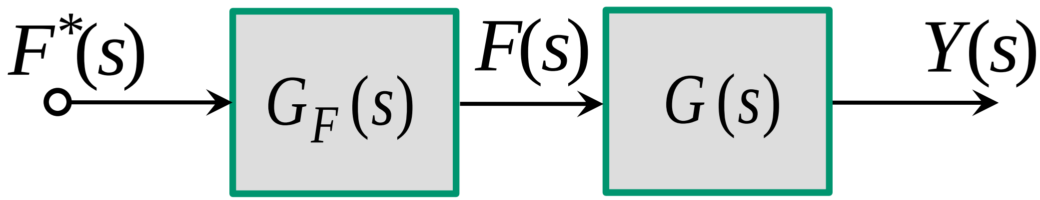
Discrete Systems Properties

- Discrete-Time to Continuous-Time – Filters

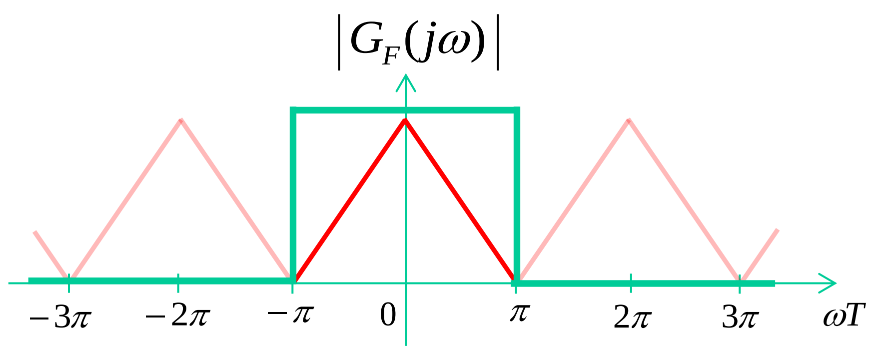


Discrete Systems Properties

- Discrete-Time to Continuous-Time – Filters

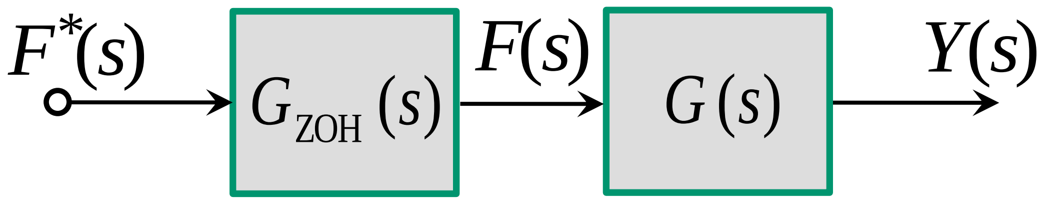


$$F(s) = G_F(s)F^*(s)$$

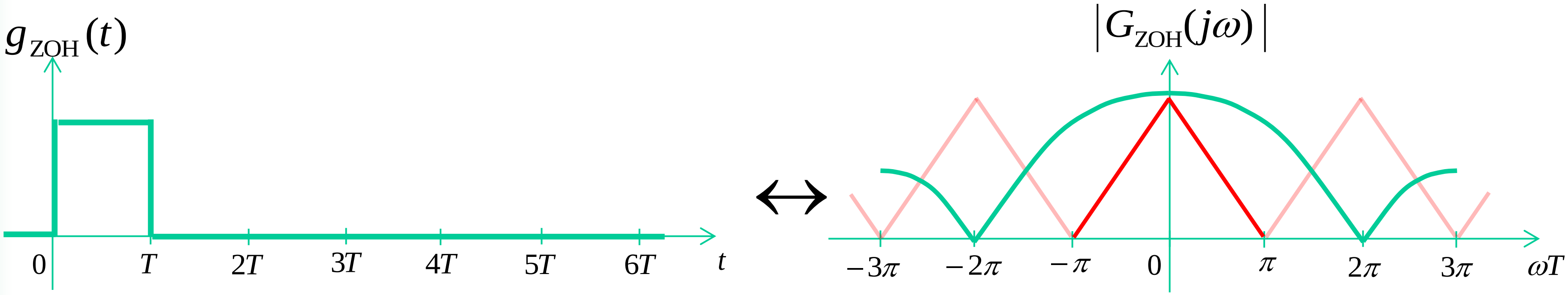


Discrete Systems Properties

- Discrete-Time to Continuous-Time – ZOH

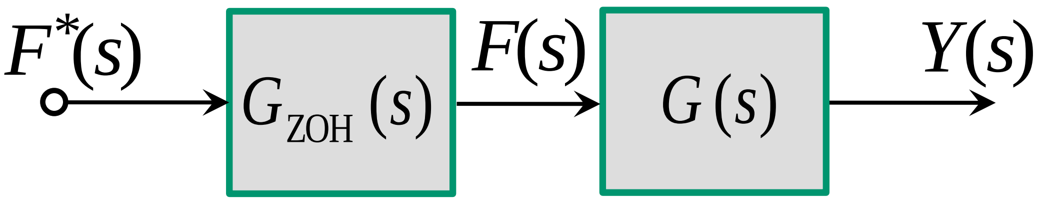


$$G_{ZOH}(s) = \frac{1 - e^{-sT}}{s}$$

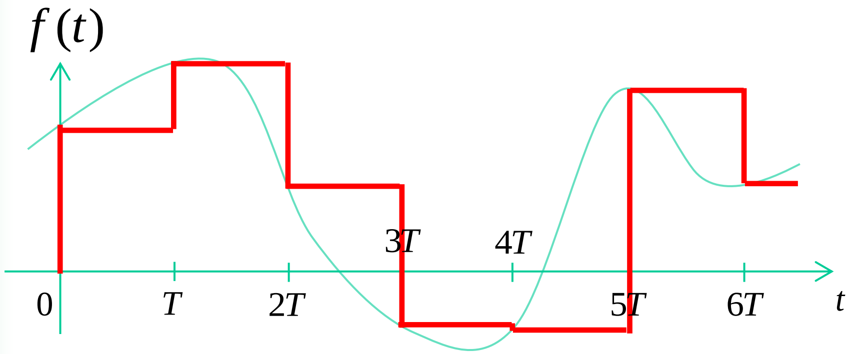


Discrete Systems Properties

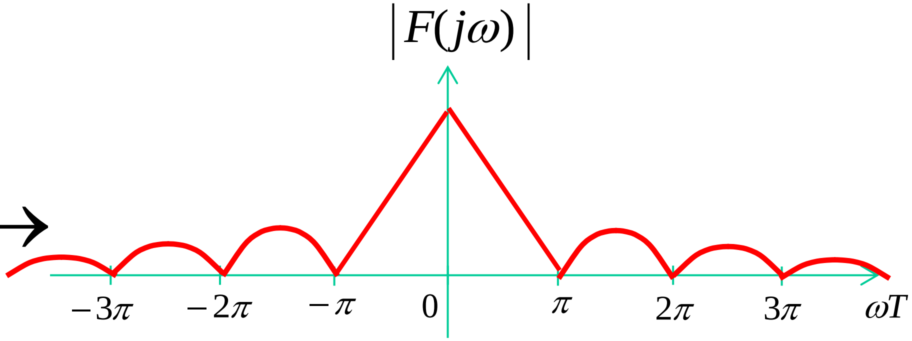
- Discrete-Time to Continuous-Time – ZOH



$$G_{\text{ZOH}}(s) = \frac{1 - e^{-sT}}{s}$$



$\leftrightarrow$



Discrete Systems Properties

- Example (Phillips & Nagle, Example 8.1)
  - Plant, $G_p(s) = \dfrac{8}{s(s+1)(\frac{1}{2}s+1)}$.
  - Selection of T:

    System's fastest time constant is 0.5 s. A *rule of thumb* is to choose T one-tenth of the fastest time constant i.e. $T = 0.05$ s.

Discrete Systems Properties

- Example cont'd
  - ZOH–Plant combination

$$G(s) = \underbrace{\left(1 - e^{-sT}\right)}_{G_1(s)} \underbrace{\frac{1}{s^2(s+1)(0.5s+1)}}_{G_2(s)}$$

  - Discrete part of ZOH

$$G_1(s) = 1 - e^{-0.05s} \qquad (T = 0.05\text{s})$$

$$G_1(z) = \mathcal{Z}\left[1 - e^{-0.05s}\right] = 1 - z^{-1} = \frac{z-1}{z}$$

Discrete Systems Properties

- Example cont'd
  - ZOH–Plant combination

$$G(s) = \underbrace{\left(1 - e^{-sT}\right)}_{G_1(s)} \underbrace{\frac{1}{s^2 (s+1)(0.5s+1)}}_{G_2(s)}$$

  - Discrete part of ZOH

$$G_1(s) = 1 - e^{-0.05s} \qquad (T = 0.05\text{s})$$

$$G_1(z) = \mathcal{Z}\left[1 - e^{-0.05s}\right] = 1 - z^{-1} = \frac{z-1}{z}$$

Discrete Systems Properties

- Example cont'd
  - Continuous part of ZOH and plant

$$G_2(s) = \frac{1}{s^2(s+1)(0.5s+1)} = \frac{1}{s^2} - \frac{1.5}{s} + \frac{2}{s+1} - \frac{0.5}{s+2}$$

  - Taking the Z-transform then yields

$$G_2(z) = \mathcal{Z}\left[\frac{1}{s^2} - \frac{1.5}{s} + \frac{2}{s+1} - \frac{0.5}{s+2}\right]$$

$$= \frac{0.005z}{(z-1)^2} - \frac{1.5z}{z-1} + \frac{2z}{z-0.9512} - \frac{0.5z}{z-0.9048}$$

Discrete Systems Properties

- Example cont'd
  - ZOH & plant combination pulse transfer function becomes

$$G(z) = G_1(z)G_2(z) =$$

$$= \frac{z-1}{z}\left[\frac{0.005z}{(z-1)^2} - \frac{1.5z}{z-1} + \frac{2z}{z-0.9512} - \frac{0.5z}{z-0.9048}\right]$$

$$= \frac{-0.045z^5 + 0.1738z^4 - 0.2512z^3 + 0.1612z^2 - 0.03873z}{z^6 - 4.856z^5 + 9.429z^4 - 9.15z^3 + 4.438z^2 - 0.8606z}$$

Tutorial Exercises & Homework

- Tutorial Exercises
  - Derive the frequency response of a ZOH with a sampling period of T s.
  - A First-Order Hold (FOH) has the transfer function

    $$G_{FOH}(s) = G_{ZOH}^2(s).$$

    Show that it is not causal and find the time shift (delay) making it causal.
- Homework
  - Study all relevant sections in Burns.

Conclusion

- Transient Response
- Stability
- Root locus
- Discrete-time to continuous-time conversion
  - Filters and Zero-Order Hold (ZOH)
- Tutorial Exercises & Homework

Next Attraction! – Miss It & You'll Miss Out!

- Digital Control System Design (Burns, Chapter 7)

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Thank you
for your interest!