CONTROL I

ELEN3016

Discrete-Time Signals & Systems
and
Z-Transform

(Lecture 22)

Overview

- First Things First!
- Signals and Sampling
- Z-Transforms
- Systems and Sampling
- Digital Controller Design

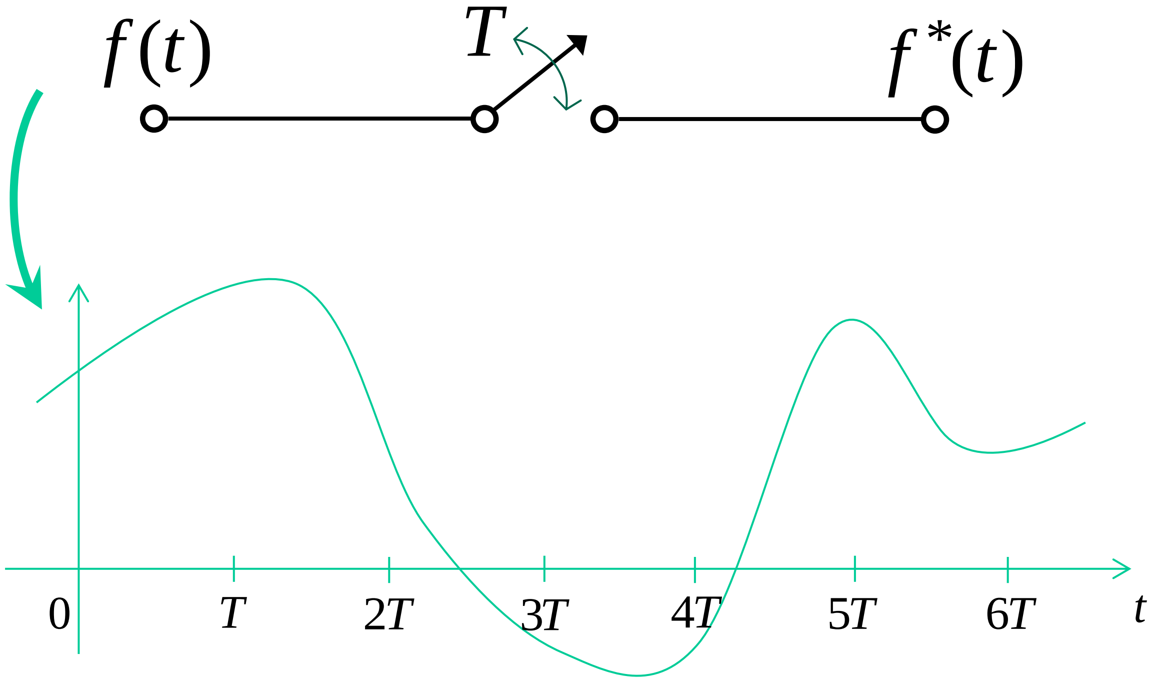
  (Classical Methods, Numerical Methods, Analytical Design, Optimum Response Digital Design)
- Tutorial Exercises & Homework

First Things First!

- Thursday 22 September 2012 public holiday – no tut session.

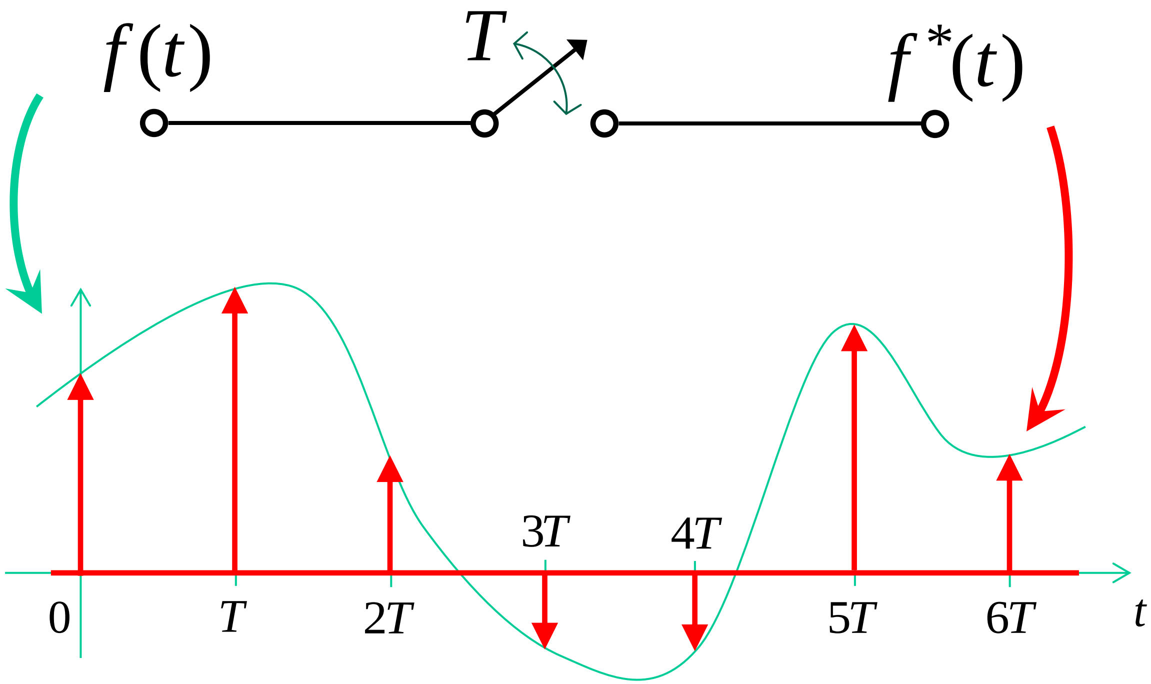
Signals and Sampling

- Signal Sampling



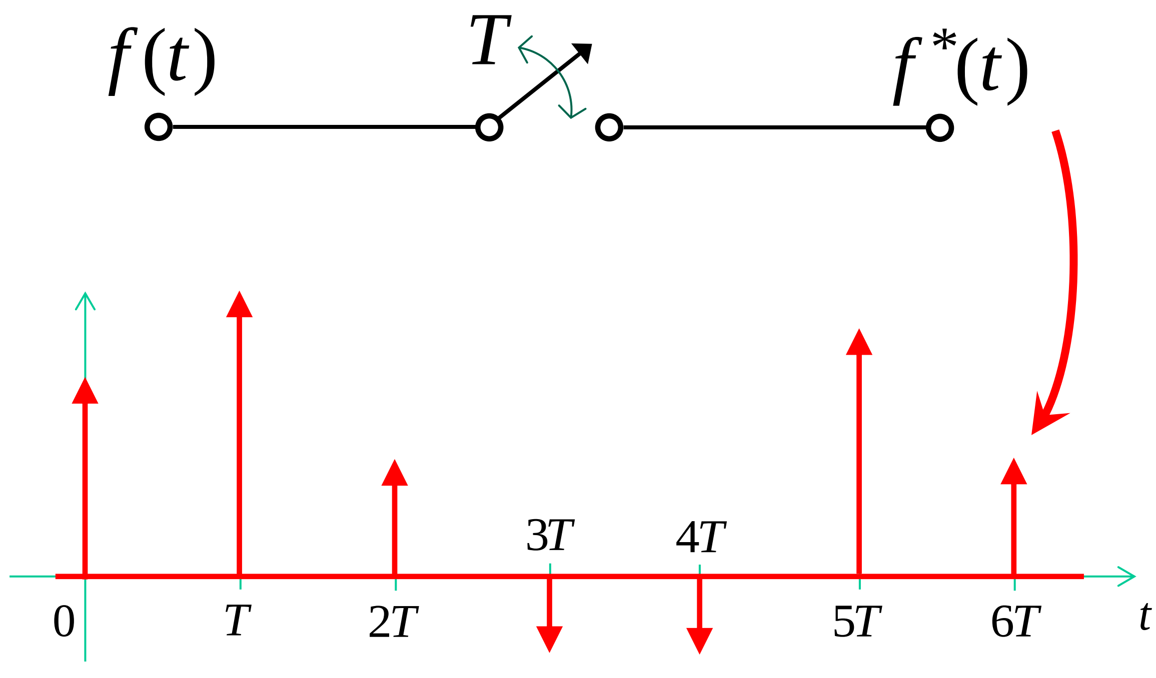
Signals and Sampling

- Signal Sampling cont'd



Signals and Sampling

- Signal Sampling cont'd



Signals and Sampling

- Signal Sampling cont'd

Sampled signal as a convolution sum

$$f^*(t) = \sum_{n=0}^{\infty} f(nT)\delta(t - nT)$$

Taking the Laplace transform yields

$$F^*(s) = \mathfrak{L}\left(f^*(t)\right) = \sum_{n=0}^{\infty} f(nT)e^{-snT}$$

Z-Transform: $F(z) = \mathfrak{Z}\left(f(t)\right) = \mathfrak{L}\left(f^*(t)\right)_{e^{sT} \mapsto z}$

Z-Transform

- Some Important Z-Transforms

1. $\mathfrak{Z}\left[a^n\right] = \sum_{k=0}^{\infty} a^k z^{-k} = \sum_{k=0}^{\infty}\left(\frac{a}{z}\right)^k = \frac{1}{1-\frac{a}{z}} = \frac{z}{z-a}$

2. $\mathfrak{Z}\left[na^n\right] = \mathfrak{Z}\left[a\frac{d}{da}(a^n)\right] = a\frac{d}{da}\mathfrak{Z}\left[a^n\right] = \frac{az}{(z-a)^2}$

3. $\mathfrak{Z}[n] = \mathfrak{Z}\left[\left. na^n\right|_{a=1}\right] = \mathfrak{Z}\left[na^n\right]_{a=1} = \frac{z}{(z-1)^2}$

4. $\mathfrak{Z}\left[e^{j\omega n}\right] = \mathfrak{Z}\left[\left. a^n\right|_{a=e^{j\omega}}\right] = \mathfrak{Z}\left[a^n\right]_{a=e^{j\omega}} = \frac{z}{z-e^{j\omega}}$

Z-Transform

- Some Important Z-Transforms

1. $\mathfrak{Z}\left[a^n\right]=\sum_{k=0}^{\infty} a^k z^{-k}=\sum_{k=0}^{\infty}\left(\frac{a}{z}\right)^k=\frac{1}{1-\frac{a}{z}}=\frac{z}{z-a}$
2. $\mathfrak{Z}\left[na^n\right]=\mathfrak{Z}\left[a\frac{d}{da}(a^n)\right]=a\frac{d}{da}\mathfrak{Z}\left[a^n\right]=\frac{az}{(z-a)^2}$
3. $\mathfrak{Z}[n]=\mathfrak{Z}\left[\left. na^n\right|_{a=1}\right]=\mathfrak{Z}\left[na^n\right]_{a=1}=\frac{z}{(z-1)^2}$
4. $\mathfrak{Z}\left[e^{j\omega n}\right]=\mathfrak{Z}\left[\left. a^n\right|_{a=e^{j\omega}}\right]=\mathfrak{Z}\left[a^n\right]_{a=e^{j\omega}}=\frac{z}{z-e^{j\omega}}$

Z-Transform

- Some Important Z-Transforms

1. $\mathfrak{Z}\left[a^n\right] = \sum_{k=0}^{\infty} a^k z^{-k} = \sum_{k=0}^{\infty} \left(\frac{a}{z}\right)^k = \frac{1}{1-\frac{a}{z}} = \frac{z}{z-a}$

2. $\mathfrak{Z}\left[na^n\right] = \mathfrak{Z}\left[a\frac{d}{da}(a^n)\right] = a\frac{d}{da}\mathfrak{Z}\left[a^n\right] = \frac{az}{(z-a)^2}$

3. $\mathfrak{Z}[n] = \mathfrak{Z}\left[\left. na^n\right|_{a=1}\right] = \mathfrak{Z}\left[na^n\right]_{a=1} = \frac{z}{(z-1)^2}$

4. $\mathfrak{Z}\left[e^{j\omega n}\right] = \mathfrak{Z}\left[\left. a^n\right|_{a=e^{j\omega}}\right] = \mathfrak{Z}\left[a^n\right]_{a=e^{j\omega}} = \frac{z}{z-e^{j\omega}}$

Z-Transform

- Some Important Z-Transforms

1. $\mathfrak{Z}\left[a^n\right] = \sum_{k=0}^{\infty} a^k z^{-k} = \sum_{k=0}^{\infty} \left(\frac{a}{z}\right)^k = \frac{1}{1-\frac{a}{z}} = \frac{z}{z-a}$

2. $\mathfrak{Z}\left[na^n\right] = \mathfrak{Z}\left[a\frac{d}{da}(a^n)\right] = a\frac{d}{da}\mathfrak{Z}\left[a^n\right] = \frac{az}{(z-a)^2}$

3. $\mathfrak{Z}[n] = \mathfrak{Z}\left[\left.na^n\right|_{a=1}\right] = \mathfrak{Z}\left[na^n\right]_{a=1} = \frac{z}{(z-1)^2}$

4. $\mathfrak{Z}\left[e^{j\omega n}\right] = \mathfrak{Z}\left[\left.a^n\right|_{a=e^{j\omega}}\right] = \mathfrak{Z}\left[a^n\right]_{a=e^{j\omega}} = \frac{z}{z-e^{j\omega}}$

Z-Transform

- Some Important Z-Transforms cont'd

5. $\Im[\sin\omega n] = \frac{1}{2j}\Im\left[e^{j\omega n} - e^{-j\omega n}\right] = \ldots = \frac{z\sin\omega}{z^2 - 2z\cos\omega + 1}$

6. $\Im[\delta(n)] = \Im\left[\lim_{a\to 0} a^n\right] = \lim_{a\to 0}\Im\left[a^n\right] = 1$

7. $\Im[u(n)] = \Im\left[\left. a^n\right|_{a=1}\right] = \Im\left[a^n\right]_{a=1} = \frac{z}{z-1}$

Conclusion: Many transforms follow from that of the standard exponential function in 1.) above.

Z-Transform

- Some Important Z-Transforms cont'd

5. $\mathcal{Z}[\sin \omega n] = \frac{1}{2j}\mathcal{Z}\left[e^{j\omega n} - e^{-j\omega n}\right] = \ldots = \frac{z\sin\omega}{z^2 - 2z\cos\omega + 1}$

6. $\mathcal{Z}[\delta(n)] = \mathcal{Z}\left[\lim_{a\to 0} a^n\right] = \lim_{a\to 0} \mathcal{Z}\left[a^n\right] = 1$

7. $\mathcal{Z}[u(n)] = \mathcal{Z}\left[\left. a^n \right|_{a=1}\right] = \mathcal{Z}\left[a^n\right]_{a=1} = \frac{z}{z-1}$

Conclusion: Many transforms follow from that of the standard exponential function in 1.) above.

Z-Transform

- Some Important Z-Transforms cont'd

5. $\mathfrak{Z}[\sin \omega n] = \frac{1}{2j}\mathfrak{Z}\left[e^{j\omega n} - e^{-j\omega n}\right] = \ldots = \frac{z\sin\omega}{z^2 - 2z\cos\omega + 1}$

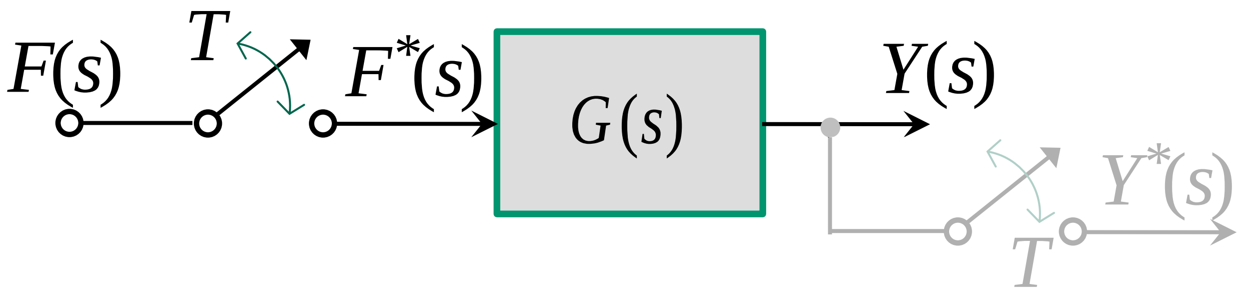
6. $\mathfrak{Z}[\delta(n)] = \mathfrak{Z}\left[\lim_{a\to 0} a^n\right] = \lim_{a\to 0}\mathfrak{Z}\left[a^n\right] = 1$

7. $\mathfrak{Z}[u(n)] = \mathfrak{Z}\left[\left. a^n \right|_{a=1}\right] = \mathfrak{Z}\left[a^n\right]_{a=1} = \frac{z}{z-1}$

Conclusion: Many transforms follow from that of the exponential function in 1.) above.

Systems and Sampling

- Block Diagram Algebra – Example 1



For the above system we have

$$y(t) = \sum_{k=0}^{\infty} f(kT)g(t-kT) \qquad \text{(Signals \& Systems)}$$

and

$$y^*(t) = \sum_{n=0}^{\infty} y(nT)\delta(t-nT)$$

Systems and Sampling

- Block Diagram Algebra – Example 1 cont'd

$$y^*(t)=\sum_{n=0}^{\infty}\sum_{k=0}^{\infty} f(kT)g(nT-kT)\delta(t-nT)$$

Taking the Laplace transform yields

$$Y^*(s)=\sum_{n=0}^{\infty}\sum_{k=0}^{\infty} f(kT)g(nT-kT)e^{-snT}$$

$$=\sum_{k=0}^{\infty} f(kT)\sum_{n=0}^{\infty} g(nT-kT)e^{-snT}$$

Systems and Sampling

- Block Diagram Algebra – Example 1 cont'd

$$Y^*(s) = \sum_{k=0}^{\infty} f(kT) G^*(s) e^{-skT}$$

$$= G^*(s) \left(\sum_{k=0}^{\infty} f(kT) e^{-skT} \right)$$

$$= G^*(s) F^*(s)$$

Letting $e^{sT} \mapsto z$ yields $Y(z) = G(z) F(z)$.

Systems and Sampling

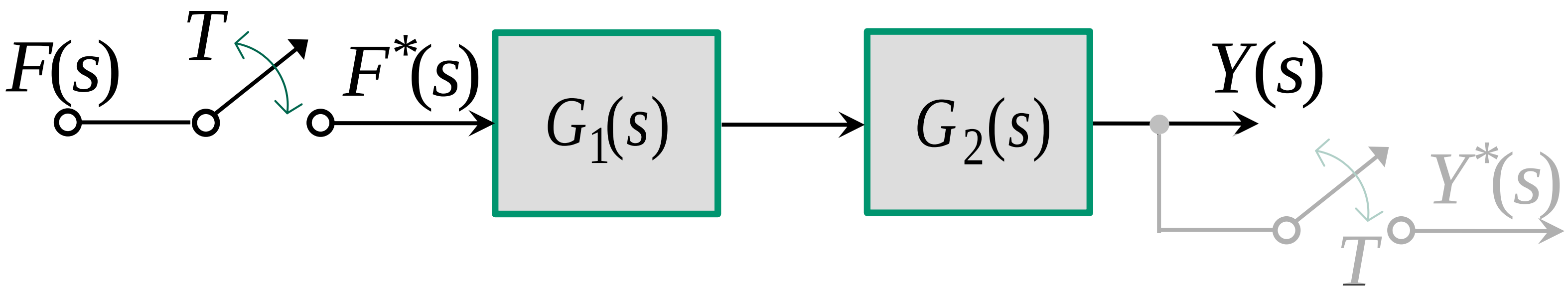
- Block Diagram Algebra cont'd

From the above we obtain the following algebraic rules:

1. $\left[G^*(s)\right]^* = G^*(s)$
2. $\mathcal{L}\big(y(t)\big)^* = \mathcal{L}\Big(y^*(t)\Big)$ i.e. $\left[Y(s)\right]^* = Y^*(s)$
3. $\left[F^*(s)G(s)\right]^* = F^*(s)\left[G(s)\right]^* = F^*(s)G^*(s)$
4. Generally $\left[F(s)G(s)\right]^* \neq F^*(s)G^*(s)$

Systems and Sampling

- Block Diagram Algebra – Example 2



From the above system we have

$$Y(s) = G_1(s)G_2(s)F^*(s).$$

Applying the star-operator yields

$$Y^*(s) = \left[G_1(s)G_2(s)F^*(s)\right]^* = \left[G_1(s)G_2(s)\right]^* F^*(s).$$

Systems and Sampling

- Block Diagram Algebra – Example 2 cont'd

For ease of reference we define

$$G_1G_2^*(s) := [G_1(s)G_2(s)]^*$$

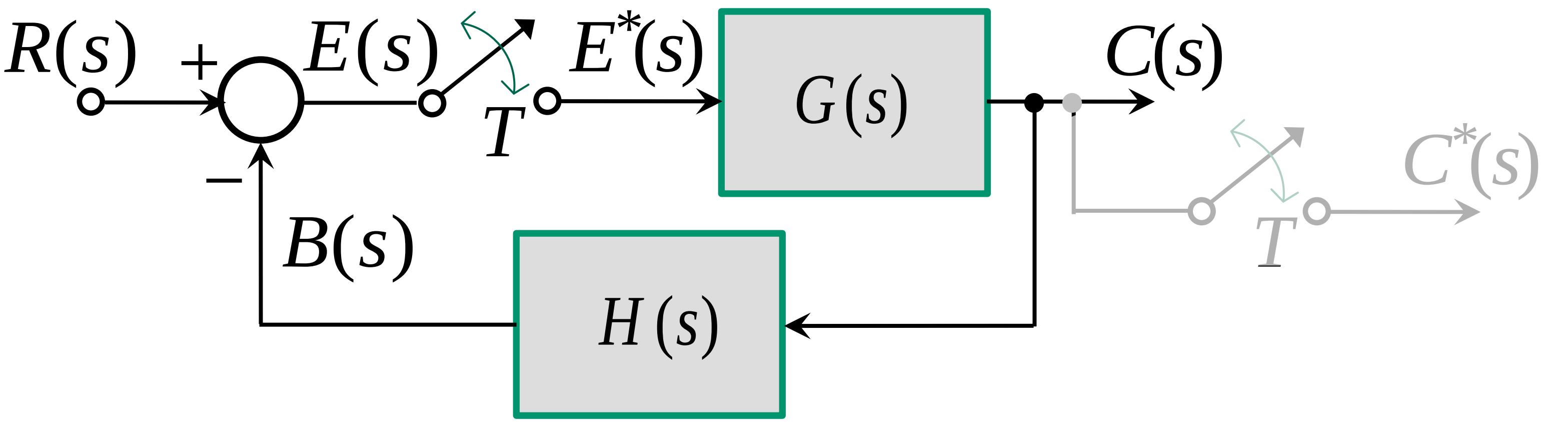
hence

$$Y^*(s) = [G_1G_2(s)]^* F^*(s)$$

Letting $e^{sT} \mapsto z$ yields $Y(z) = G_1G_2(z)\, F(z)$.

Systems and Sampling

- Block Diagram Algebra – Example 3



For the above system we have

$$C(s) = G(s)\,E^*(s) \quad \Rightarrow \quad C^*(s) = G^*(s)\,E^*(s) \quad (1)$$

Systems and Sampling

- Block Diagram Algebra – Example 3 cont'd

Furthermore

$$E(s) = R(s) - B(s)$$
$$= R(s) - H(s)C(s)$$
$$= R(s) - G(s)H(s)E^*(s).$$

Applying the star-operator yields

$$E^*(s) = R^*(s) - GH^*(s)E^*(s).$$

Systems and Sampling

- Block Diagram Algebra – Example 3 cont'd

Solving for $E^*(s)$ gives

$$E^*(s) = \frac{R^*(s)}{1+GH^*(s)}. \qquad (2)$$

Substituting (2) into (1) yields

$$C^*(s) = \frac{G^*(s)}{1+GH^*(s)} R^*(s).$$

Systems and Sampling

- Block Diagram Algebra – Example 3 cont'd

Letting $e^{sT} \mapsto z$ yields

$$
\begin{aligned}
C(z) &= C^*(s)\Big|_{e^{sT}\mapsto z} \\
&= \left[\frac{G^*(s)}{1+GH^*(s)}R^*(s)\right]_{e^{sT}\mapsto z} \\
&= \frac{G(z)}{1+GH(z)}R(z).
\end{aligned}
$$

Tutorial Exercises & Homework

- Tutorial Exercises
  - Prove that $G_1 G_2(z) \neq G_1(z) G_2(z)$ in general.
- Homework
  - None

Conclusion

- Z-transforms
- Signals & Sampling
- Tutorial Exercises & Homework

Next Attraction! – Miss It & You'll Miss Out!

- Discrete-Time Systems Properties (Burns, Chapter 7)

  ...

Thank you for your interest!