CONTROL I

ELEN3016

Steady-State Error Analysis

(Lecture 18)

Overview

- First Things First!
- Steady-state Error Analysis & Examples
- Tutorial Exercises & Homework

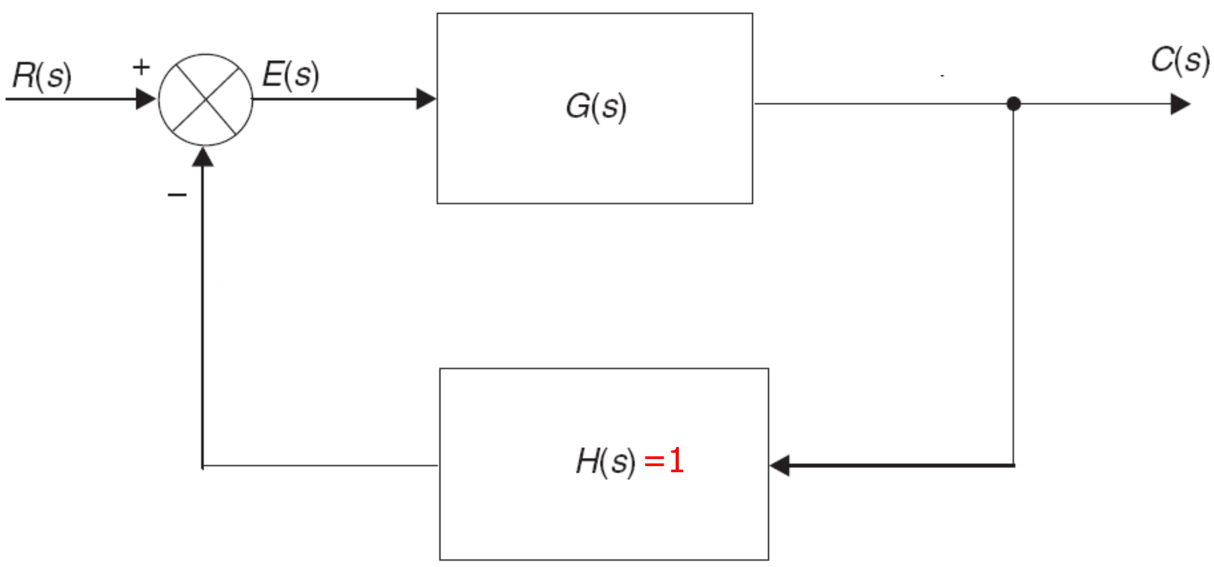
- Next Attraction!

First Things First!

- None

Steady-State Error Analysis

- Setting the Scene
  - Concentrate on the unity-feedback case.



Steady-State Error Analysis

- Setting the Scene (cont'd)
  - Error-to-output transfer function is

$$\frac{E(s)}{R(s)} = \frac{1}{1+G(s)}.$$

  - Steady-state error for unity-feedback is

$$e_{ss} = \lim_{t\to\infty} e(t) = \lim_{s\to 0} sE(s) = \lim_{s\to 0} \frac{sR(s)}{1+G(s)}.$$

Steady-State Error Analysis

- System Type for Unity-Feedback
  - *System type* is defined to be the number of poles of the system at the origin (i.e. $s = 0$).

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  - *System type* is defined to be the number of poles of the system at the origin (i.e. $s=0$).
  - Examples:

$$G(s)=\frac{(s+2)}{s^{2}(s+3)}$$ is of type 2.

$$G(s)=\frac{\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2}$$ is of type 0 for $|\zeta|<\infty$.

Steady-State Error Analysis

- Steady-State Error for Step Input
  - For a step input $R(s) = R/s$,

$$e_{ss} = \lim_{s \to 0} \frac{sR(s)}{1+G(s)} = \lim_{s \to 0} \frac{R}{1+G(s)} = \frac{R}{1+\lim_{s \to 0} G(s)}$$

Steady-State Error Analysis

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  - *Step/position error constant* is defined as

$$K_p = \lim_{s \to 0} G(s) \quad \Rightarrow \quad e_{ss}\Big|_{\text{Step}} = \frac{R}{1+K_p}$$

Steady-State Error Analysis

- Steady-State Error for Step Input (cont'd)
  - For system type 0:

$$e_{ss}\Big|_{\text{Step}} = \frac{R}{1+K_p} \neq 0 \quad \text{since} \quad |K_p| < \infty$$

  - For system type ≥ 1:

$$e_{ss}\Big|_{\text{Step}} = 0 \quad \text{since} \quad |K_p| = \infty$$

Steady-State Error Analysis

- Steady-State Error for Ramp Input
  - For a ramp input $R(s) = R/s^2$,

$$e_{ss} = \lim_{s\to 0} \frac{sR(s)}{1+G(s)} = \lim_{s\to 0} \frac{R}{s+sG(s)} = \frac{R}{\lim\limits_{s\to 0} sG(s)}$$

Steady-State Error Analysis

- Steady-State Error for Ramp Input
  - For a ramp input $R(s) = R/s^2$,

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  - *Ramp/velocity error constant* is defined as

$$K_v = \lim_{s\to 0} s\,G(s) \quad \Rightarrow \quad e_{ss}\Big|_{\text{Ramp}} = \frac{R}{K_v}$$

Steady-State Error Analysis

- Steady-State Error for Ramp Input (cont'd)
  - For system type 0: $\left. e_{ss} \right|_{\text{Ramp}} = \pm\infty$
  - For system type 1: $\left. e_{ss} \right|_{\text{Ramp}} = \dfrac{R}{K_v} = \text{const.}$
  - For system type ≥ 2: $\left. e_{ss} \right|_{\text{Ramp}} = 0$

Steady-State Error Analysis

- Steady-State Error for Parabolic Input
  - For a quadratic input $R(s) = R/s^3$,

$$e_{ss} = \lim_{s \to 0} \frac{sR(s)}{1+G(s)} = \lim_{s \to 0} \frac{R}{s^2 + s^2 G(s)} = \frac{R}{\lim_{s \to 0} s^2 G(s)}$$

Steady-State Error Analysis

- Steady-State Error for Parabolic Input
  - For a quadratic input $R(s) = R/s^3$,

$$e_{ss} = \lim_{s\to 0} \frac{sR(s)}{1+G(s)} = \lim_{s\to 0} \frac{R}{s^2 + s^2 G(s)} = \frac{R}{\lim\limits_{s\to 0} s^2 G(s)}$$

  - *Parabolic/acceleration error constant* is defined as

$$K_a = \lim_{s\to 0} s^2 G(s) \quad \Rightarrow \quad e_{ss}\Big|_{\text{Parabola}} = \frac{R}{K_a}$$

Steady-State Error Analysis

- Steady-State Error for Parabolic Input
  - For system type 0: $e_{ss}\Big|_{\text{Parabola}} = \pm\infty$
  - For system type 1: $e_{ss}\Big|_{\text{Parabola}} = \pm\infty$
  - For system type 2: $e_{ss}\Big|_{\text{Parabola}} = \dfrac{R}{K_a} = \text{const.}$
  - For system type ≥ 3: $e_{ss}\Big|_{\text{Parabola}} = 0$

Steady-State Error Analysis

- Summary
  - Error constants significant for analysis of unity-feedback systems for specific order of input.
  - Use of the FVT assumes that $sE(s)$ has no poles on $j\omega$-axis or RHP.
  - Arbitrary polynomial inputs imply a linear combination of the different orders of errors (i.e. step, ramp, parabolic etc.).
  - For non-unity feedback, follow above process.

Steady-State Error Analysis

- Example
  - The system $G(s) = \dfrac{K(s+3.15)}{s(s+1)(s+0.5)}$, $H(s) = 1$ is of type 1.
  - Step I/P: $K_p = \pm\infty$ $\quad e_{ss}\Big|_{\text{Step}} = \dfrac{R}{1+K_p} = 0$
  - Ramp I/P: $K_v = 6.3K$ $\quad e_{ss}\Big|_{\text{Ramp}} = \dfrac{R}{K_v} \approx \dfrac{0.159R}{K}$
  - Parabolic I/P: $K_a = 0$ $\quad e_{ss}\Big|_{\text{Parabola}} = \dfrac{R}{K_a} = \infty$

Steady-State Error Analysis

- Error Constants and Transfer Functions
  - For the error transfer function we have

$$\frac{E(s)}{R(s)} = \frac{1}{1+G(s)} = \frac{1}{1+K_p} + \frac{1}{K_v}s + \frac{1}{K_a}s^2 + \dots$$

yielding the closed-loop transfer function as

$$\frac{C(s)}{R(s)} = \frac{G(s)}{1+G(s)} = 1 - \frac{1}{1+K_p} - \frac{1}{K_v}s - \frac{1}{K_a}s^2 - \dots$$

Steady-State Error Analysis

- Error Constants and Poles & Zeros
  - For the closed-loop transfer function

$$\frac{C(s)}{R(s)} = \frac{G(s)}{1+G(s)} = \frac{K\prod_{i=1}^{m}(s+\tilde{z}_i)}{\prod_{j=1}^{n}(s+\tilde{p}_j)}$$

the position error constant has the form

$$K_p = \frac{K\prod_{i=1}^{m}\tilde{z}_i}{\prod_{j=1}^{n}\tilde{p}_j - K\prod_{i=1}^{m}\tilde{z}_i}.$$

Steady-State Error Analysis

- Error Constants and Poles & Zeros cont'd
  - Similarly the velocity error constant is

$$\frac{1}{K_v} = \sum_{j=1}^{n} \frac{1}{\tilde{p}_j} - \sum_{i=1}^{m} \frac{1}{\tilde{z}_i}$$

and the acceleration error constant is

$$-\frac{2}{K_a} = \frac{1}{K_v^2} + \sum_{j=1}^{n} \frac{1}{\tilde{p}_j^2} - \sum_{i=1}^{m} \frac{1}{\tilde{z}_i^2}.$$

Tutorial Exercises & Homework

- Tutorial Exercises
  - Calculate the error constants for the system
    $$G(s) = \frac{K(s+3.1)}{s^2(s+0.5)}, \quad H(s) = 1.$$
  - Prove that the velocity error constant of the proto-type 2nd order system is given by $K_v = \omega_n / 2\zeta$. (Shinners 1972, p. 171)
  - Express the error transfer as well as the closed-loop transfer functions in terms of error constants. (See earlier in these notes.)

Tutorial Exercises & Homework

- Tutorial Exercises (cont'd)
  - Express the error constants in terms of zeros and poles of the closed-loop system . (See earlier in these notes.)
- Homework
  - Study all relevant sections in Burns.

Conclusion

- Studied steady-state error analysis for unity-feedback systems.
- Tutorial Exercises & Homework

Next Attraction! – Miss It & You'll Miss Out!

- Zero-Error Systems and Controller design (Interlude)

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Thank you!

Any Questions?