CONTROL I

ELEN3016

Design using Root Locus Method

(Lecture 13)

Overview

- First Things First!
- Examples
- Root Locus Controller Design
- Variations on the Theme
- Tutorial Exercises & Homework

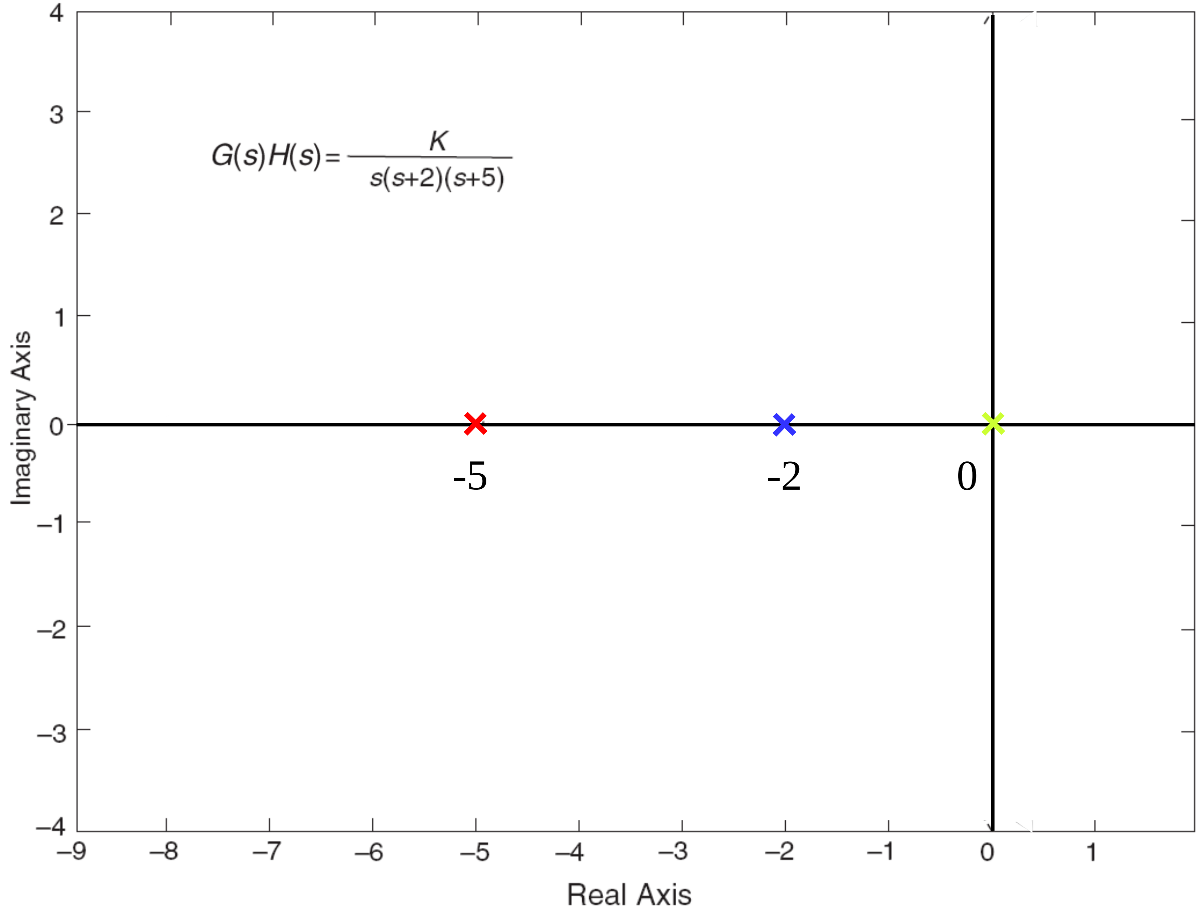
- Next Attraction!

First Things First!

- Study break … and beyond
  - Progress with Laboratory Experiment?
  - Misprint: "$G(\mathrm{s})HJ(s)$" in Figure 5.14 should read "$G(s)H(s)$".

Examples

- Example 5.8



Examples

- Open-loop
  - Open-loop transfer function: $G(s)H(s) = \dfrac{K}{s(s+2)(s+5)}$
  - Open-loop poles: $s = 0, -2, -5$
  - Open-loop zeros: none
  - Closed-loop poles (characteristic equation):

$$1 + G(s)H(s) = 0 \quad \Rightarrow \quad s^3 + 7s^2 + 10s + K = 0$$

Examples

- Starting points $\left(K=0\right)$
  - At open-loop poles: $s=0,-2,-5$
- Termination points $\left(K=\infty\right)$
  - At open-loop zeros: $m=0,\quad s=\infty e^{-j\phi_1}, \infty e^{-j\phi_2}, \infty e^{-j\phi_3}$
- No. of distinct loci
  - Equal to the degree of the Char. Eq.: $n=3$
- Asymptotes $\left(K\to\infty\right)$
  - Angles: $\alpha_k=\frac{180^\circ+k360^\circ}{3},\quad \alpha_1=60^\circ, \alpha_2=180^\circ, \alpha_3=240^\circ$

Examples

- Asymptotes' real axis intercept

$$\sigma_a = \frac{\sum_{i=1}^{m} p_i}{n-m} = \frac{0-2-5}{3} = \frac{-7}{3} = -2.333$$

- Root locus segments on the real axis
  - Segments between poles 0 & -2 and left of -5.
- Breakaway points

$$\frac{dK}{ds} = -\frac{d}{ds}\left(s^3 + 7s^2 + 10s\right) = -\left(3s^2 + 14s + 10\right) = 0$$

$$\sigma_b = \frac{-14 \pm \sqrt{14^2 - 4\times 3\times 10}}{6} = \frac{-7 \pm \sqrt{19}}{3} = -3.7863,\ -0.8804$$

Examples

- Gain at marginal stability

Characteristic Equation: $s^3 + 7s^2 + 10s + K = 0$

Routh array:

$$
\begin{array}{c|cc}
s^0 & K & 0 \\
s^1 & (70 - K)/7 & 0 \\
s^2 & 7 & K \\
s^3 & 1 & 10 \\
\hline
\end{array}
$$

Thus $K = 0$ or $K = 70$.

Examples

- Oscillation frequency at marginal stability

Where is the imaginary axis intercept for $K = 70$?

Routh array:

$$
\begin{array}{c|cc}
s^0 & 70 & 0 \\
s^1 & 0 & 0 \\
s^2 & 7 & 70 \\
s^3 & 1 & 10 \\
\hline
\end{array}
$$

Examples

- Oscillation frequency at marginal stability

Where is the imaginary axis intercept for $K = 70$?

Routh array:

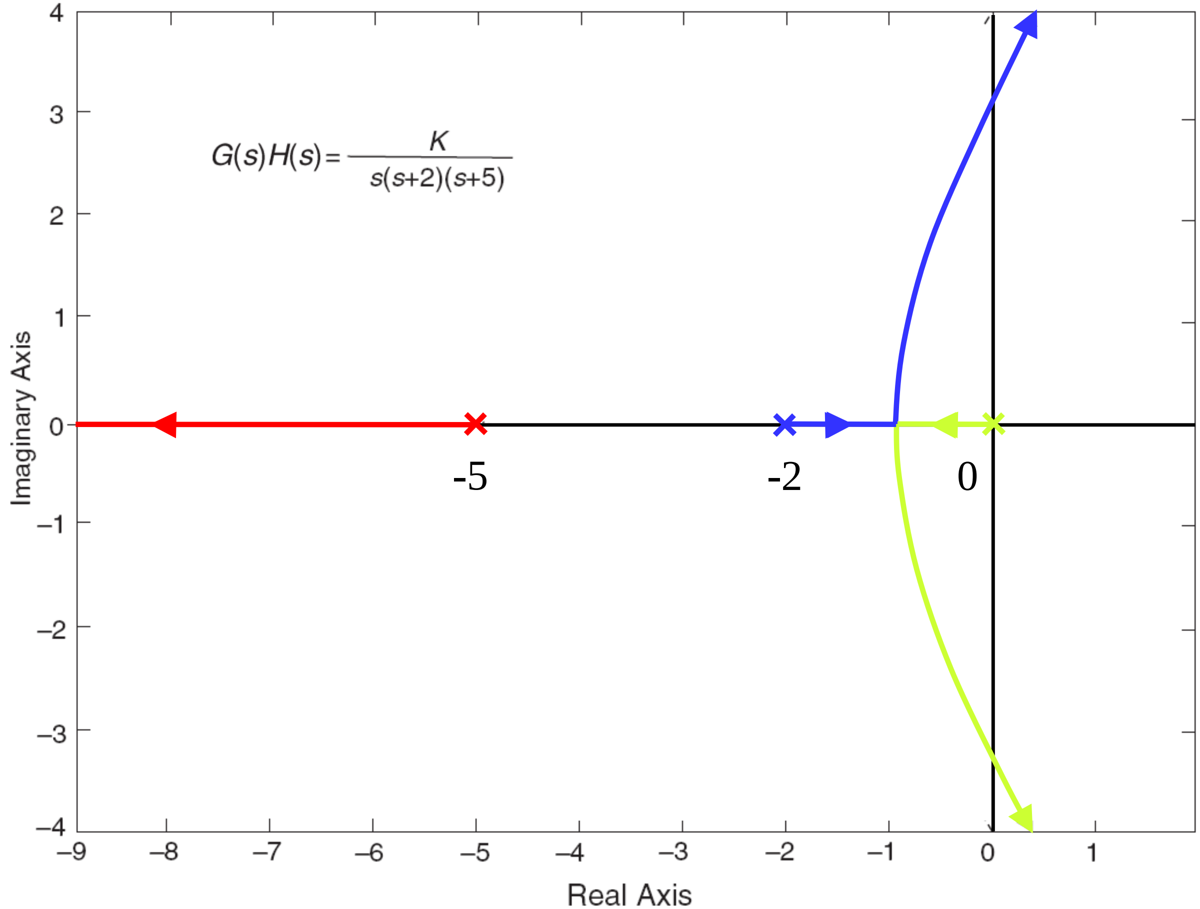
$$
\begin{array}{c|cc}
s^0 & 70 & 0 \\
s^1 & 0 & 0 \\
s^2 & 7 & 70 \\
s^3 & 1 & 10 \\
\hline
\end{array}
$$

Auxiliary equation:

$$7s^2 + 0s + 70 = 0 \;\Rightarrow\; s = \pm j\sqrt{10} \text{ rad/s}$$

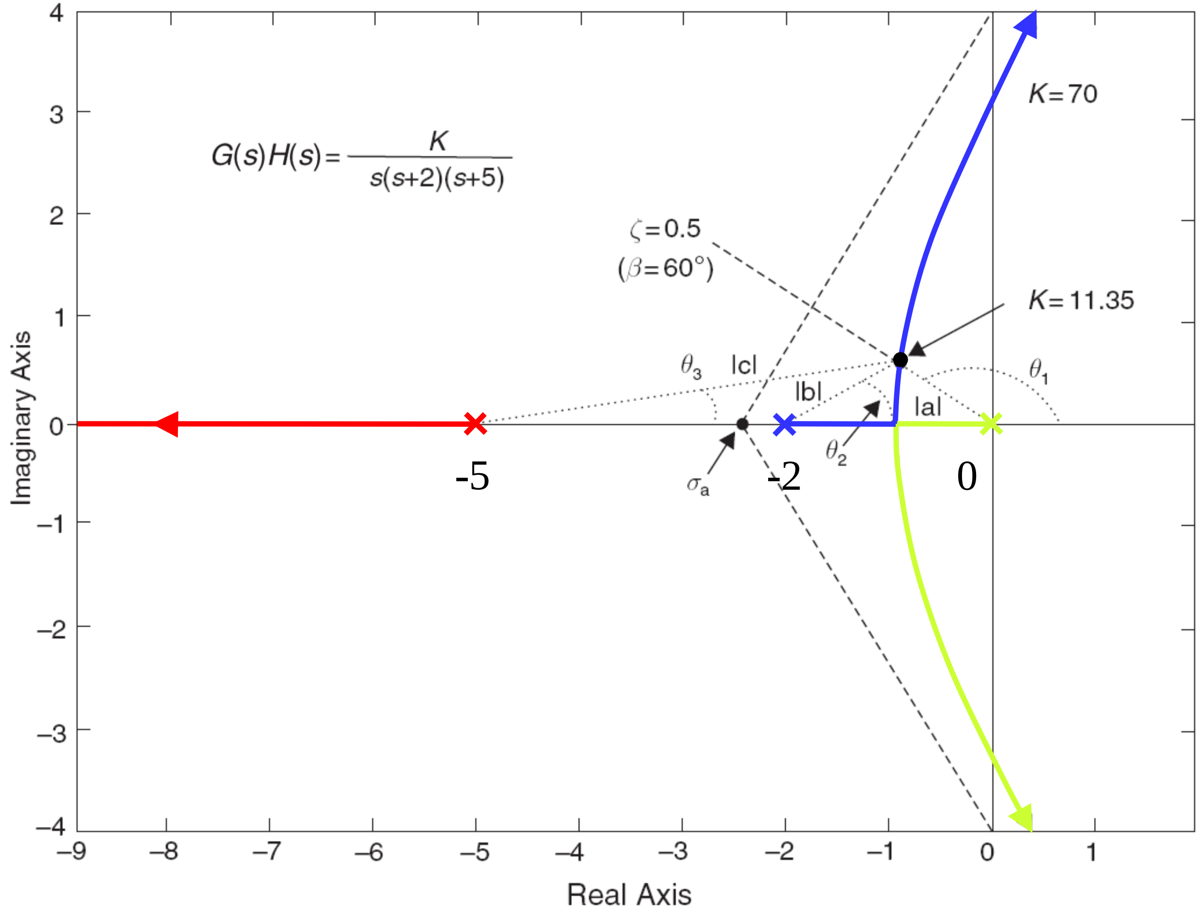
Examples

- Example 5.8 (cont'd)



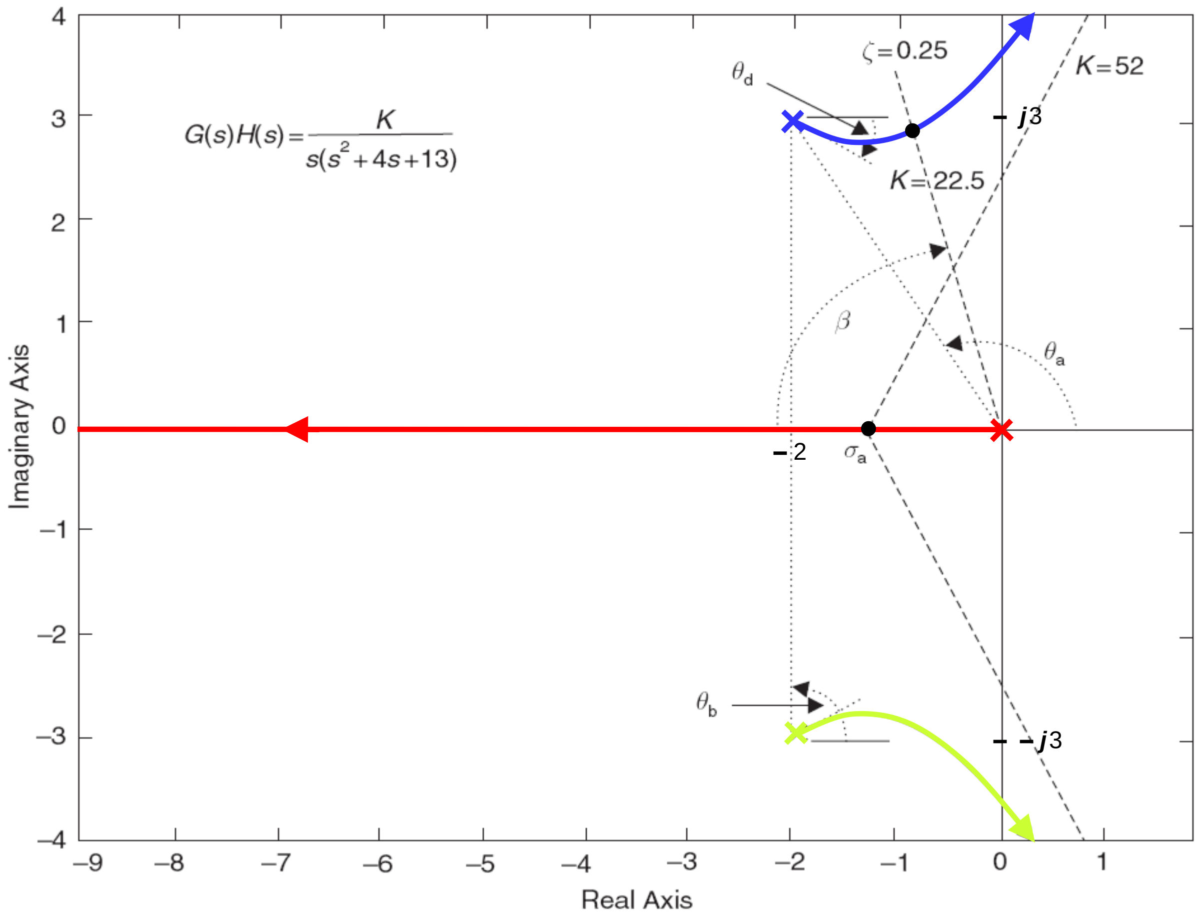
Examples

- Example 5.8 (cont'd)



Examples

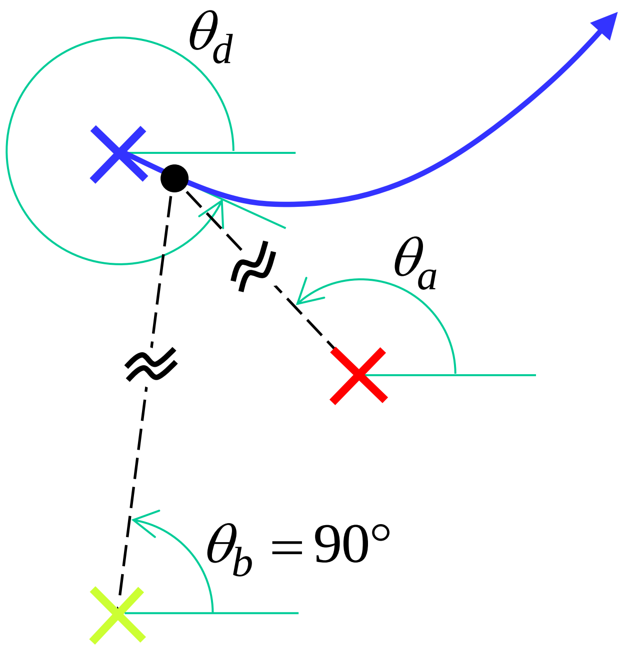
- Example 5.9



Examples

- Example 5.9 (Explanation)

  Angle of departure θ_d from the pole at $-2+j3$?

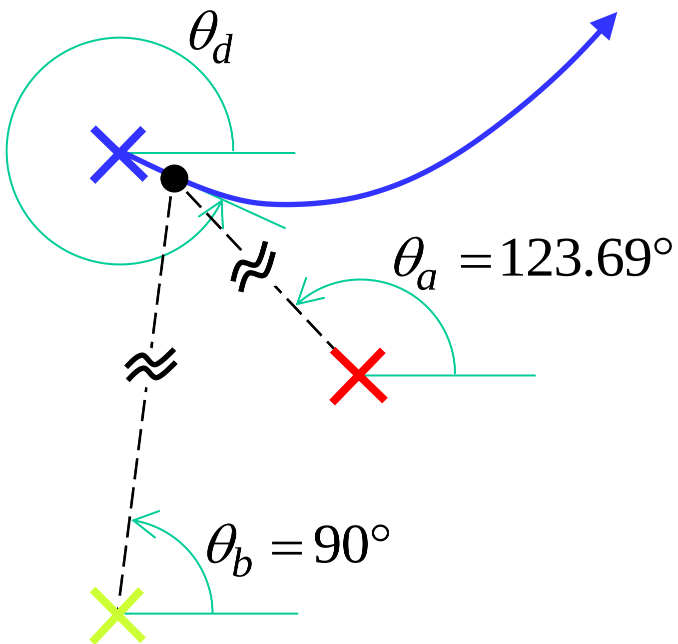


… select a point <u>on</u> the locus very close to the pole $-2+j3$ (see the black dot).

Examples

- Example 5.9 (Explanation)

Angle of departure θ_d from the pole at $-2+j3$?



Calculating θ_a:

$$\tan\left(180° - \theta_a\right) = \frac{3}{2}$$

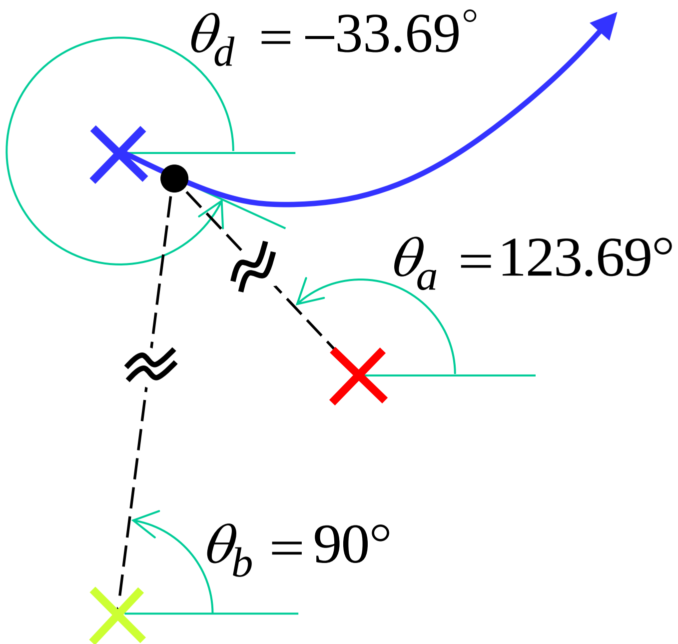
$$\theta_a = 180° - \arctan\left(\frac{3}{2}\right)$$

$$= 123.69°$$

Examples

- Example 5.9 (Explanation)

Angle of departure θ_d from the pole at $-2+j3$?



Angle criterion:

$$\theta_d + \theta_a + \theta_b = 180^\circ$$

$$\begin{aligned}\theta_d &= 180^\circ - \theta_a - \theta_b \\ &= 180^\circ - 123.69^\circ - 90^\circ \\ &= -33.69^\circ\end{aligned}$$

Root Locus Controller Design

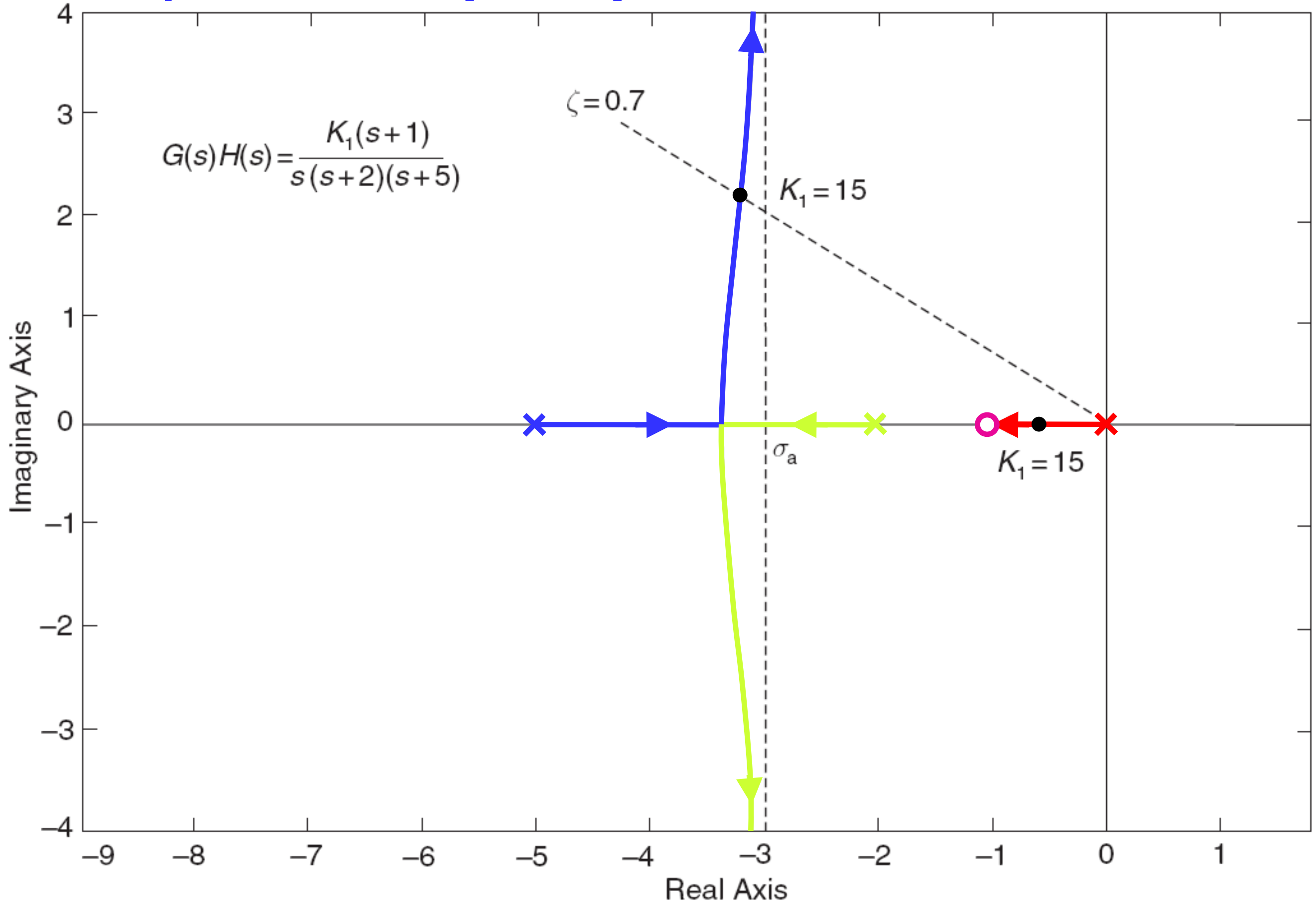
- Example 5.10 (Self-Study)
  - Open-loop: $G(s)H(s) = \left.\dfrac{K}{s(s+2)(s+5)}\right|_{K=1}$
  - PD Controller: $G_c(s) = K_1(s+a)$
  - Characteristic equation:

$$1+\underbrace{G_c(s)G(s)H(s)}_{\text{Textbook: } G(s)H(s)} = 1+\frac{K_1(s+a)}{s(s+2)(s+5)}$$

  - Specifications: PO < 5% and t_s < 2 sec

Root Locus Controller Design

- Example 5.10 (a=1)

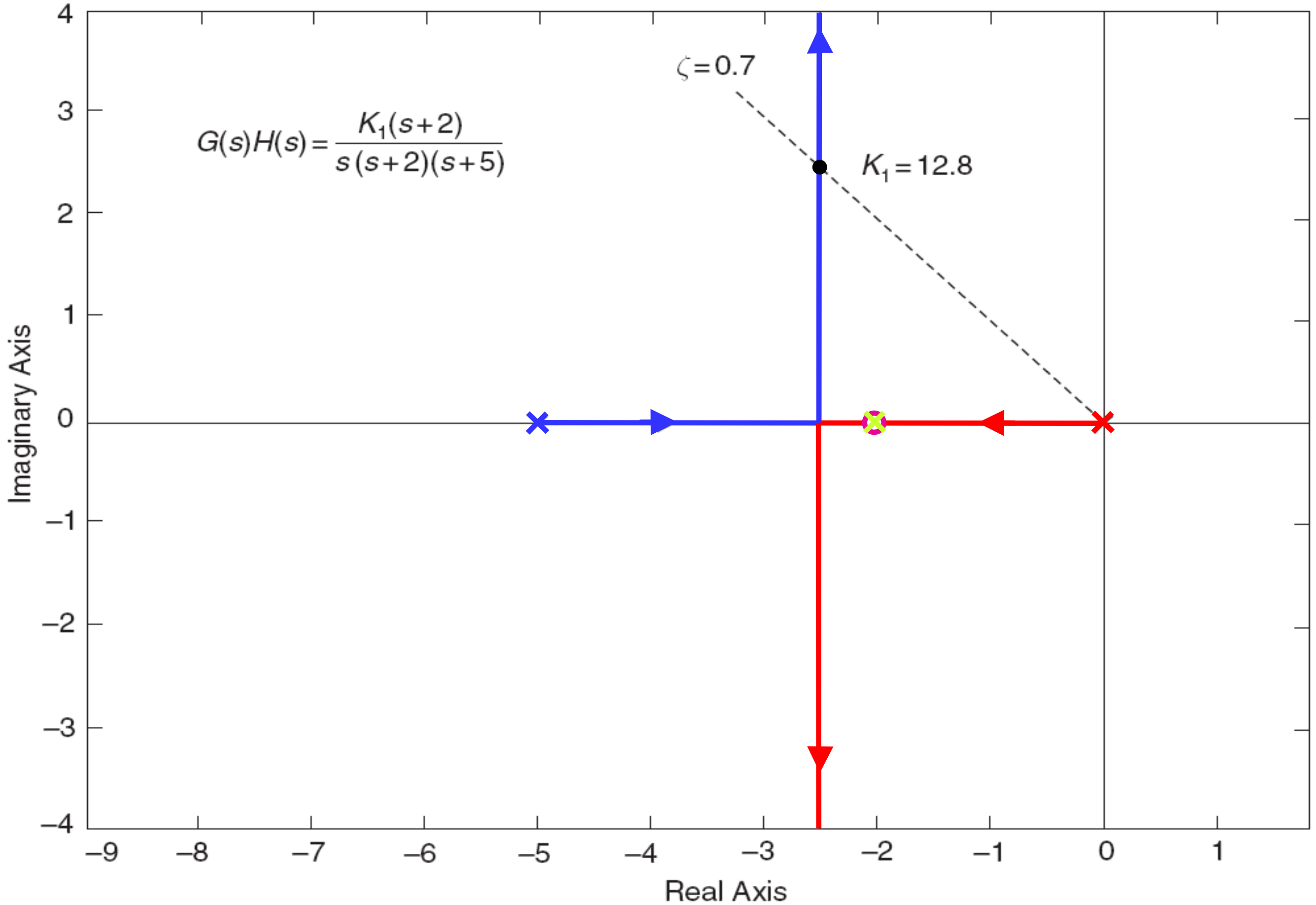


PO = 3.9%

t_s = 4 sec

Root Locus Controller Design

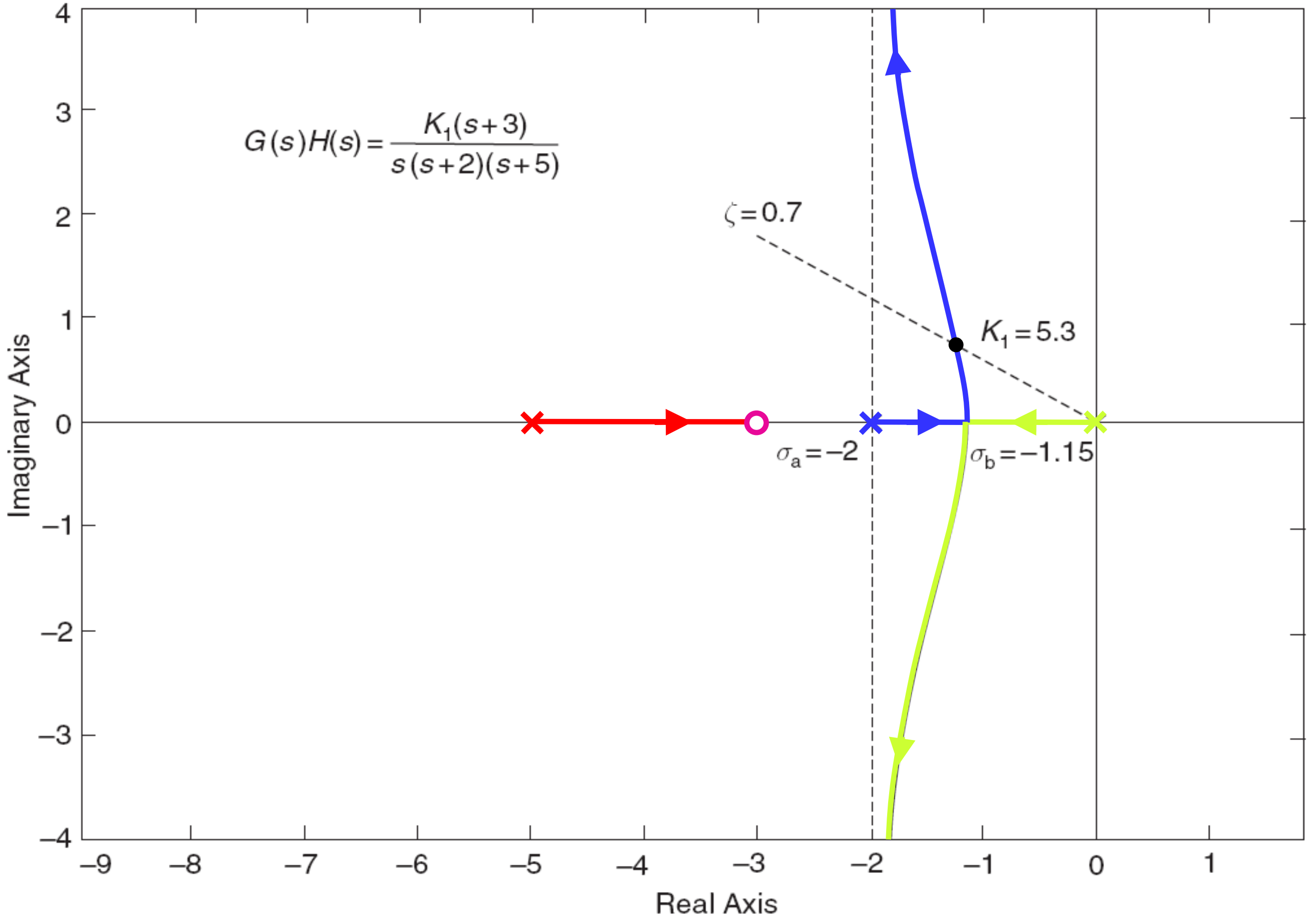
- Example 5.10 (a=2)



PO = 4.1%
t_s = 1.7 sec

Root Locus Controller Design

- Example 5.10 (a=3)

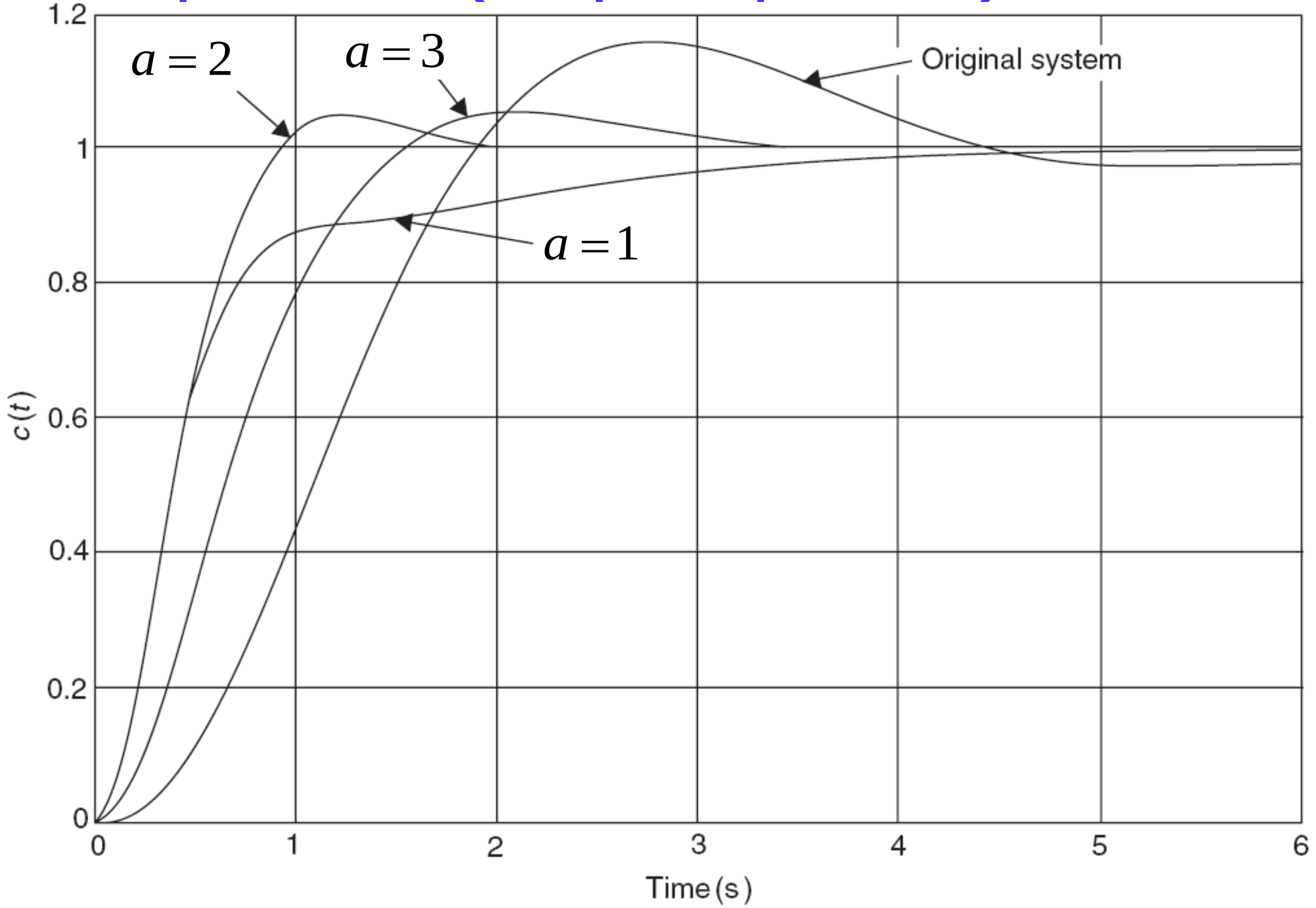


PO = 5.3%

t_s = 3.1 sec

Root Locus Controller Design

- Example 5.10 (Step responses)



Variations on the RH-Theme

- No. of poles to the right of other vertical lines
  - To find the no. of poles to the right of the vertical line $s = \sigma_0$ substitute $(s - \sigma_0)$ for s into the characteristic equation.
  - Manipulate this into a polynomial in s.
  - The no. of sign changes in the Routh Array for this new polynomial yields the no. of poles to the right of the vertical line $s = \sigma_0$.

Variations on the RH/RL-Theme

- Closed-loop stability w.r.t. other parameters
  - For any parameter a in the characteristic equation manipulate the characteristic equation into the form

$$1+a\frac{\tilde{N}(s)}{\tilde{D}(s)}=0 \qquad \Rightarrow \qquad \tilde{D}(s)+a\tilde{N}(s)=0$$

  - This is exactly in the form of the original character-istic equation but now with a as the variable gain.
  - Sign changes in the 1st column of the Routh array yields the no. of poles in the RHP due to changes in the value of a.

Tutorial & Homework Exercises

- Tutorial Exercises
  - Burns, Example 5.16
  - Sketch the root locus for the system with char. eq.
    a. $s\left(s^2+12s+45\right)+K=0,$
    b. $s\left(s+p\right)+K=0$ with $p=4+a,\ K=20$
    where $a \geq 0$ accounts for variation in the pole.
- Homework
  - Burns, Examples 5.9 and 5.11 & all relevant sections.

Conclusion

- Examples
- Root Locus Controller Design
- Variations on the Theme of the Root Locus
- Burns, Case study (Example 5.11) (Self-study!)
- Tutorial Exercises & Homework

Next Attraction! – Miss It & You'll Miss Out!

- Classical Design in the Frequency Domain (Burns, Chapter 6)

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Thank you!

Any Questions?