CONTROL I

ELEN3016

The Root Locus Method

(Lecture 12)

Overview

- First Things First!
- Introductory Examples
- Construction Rules
- Tutorial Exercises & Homework

- Next Attraction!

First Things First!

- Temporary arrangement – online access

  http://dept.eie.wits.ac.za/~vanwyk/ELEN3016_2012/

- Deadline for Labs
  - Lab assessment interviews need to start on 19 October'15.

Complex Numbers – Revision

- Algebra of Complex Numbers
  - Modulus-argument form: $z = |z| e^{j\angle z} \equiv |z| \angle z$
  - Product of complex numbers:

$$(s-p)\times(s-z) = \left(|s-p| e^{j\angle(s-p)}\right)\times\left(|s-z| e^{j\angle(s-z)}\right)$$

$$= |s-p||s-z| e^{j\angle(s-p)} e^{j\angle(s-z)}$$

$$= |s-p||s-z| e^{j[\angle(s-p)+\angle(s-z)]}$$

Complex Numbers – Revision

- Algebra of Complex Numbers (cont'd)
  - Quotient of complex numbers:

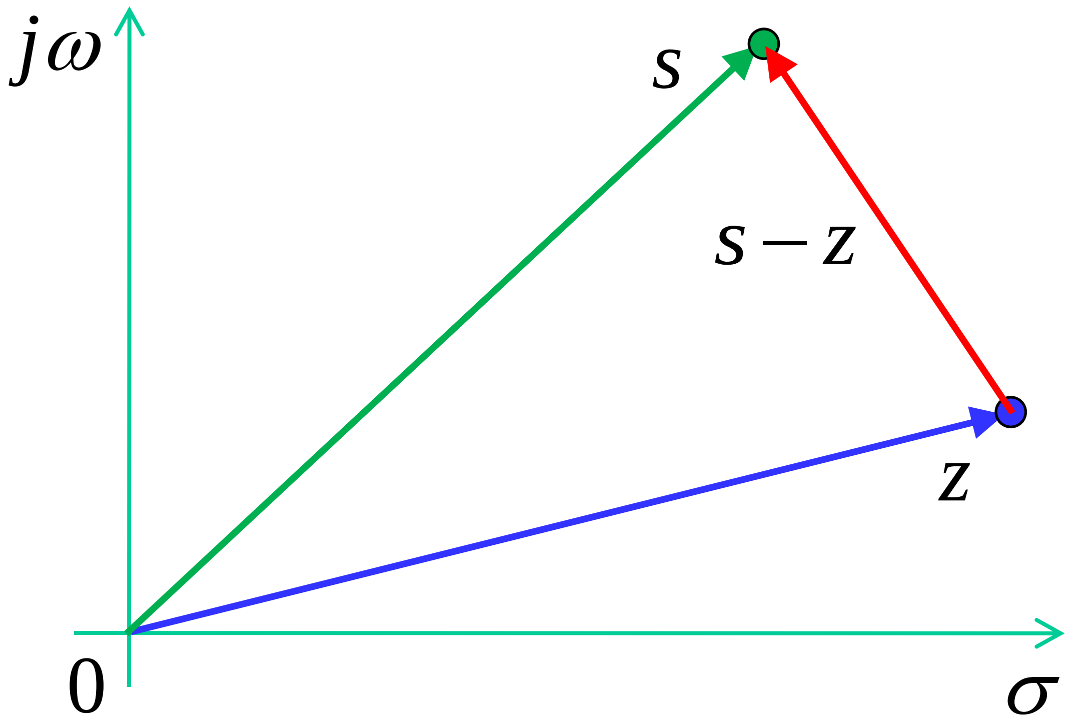
$$\frac{s-p}{s-z}=\frac{|s-p|e^{j\angle(s-p)}}{|s-z|e^{j\angle(s-z)}}$$

$$=\frac{|s-p|}{|s-z|}\times\frac{e^{j\angle(s-p)}}{e^{j\angle(s-z)}}$$

$$=\frac{|s-p|}{|s-z|}e^{j[\angle(s-p)-\angle(s-z)]}$$

Complex Numbers – Revision

- Algebra of Complex Numbers (cont'd)
  - Visualisation of $(s-z)$:



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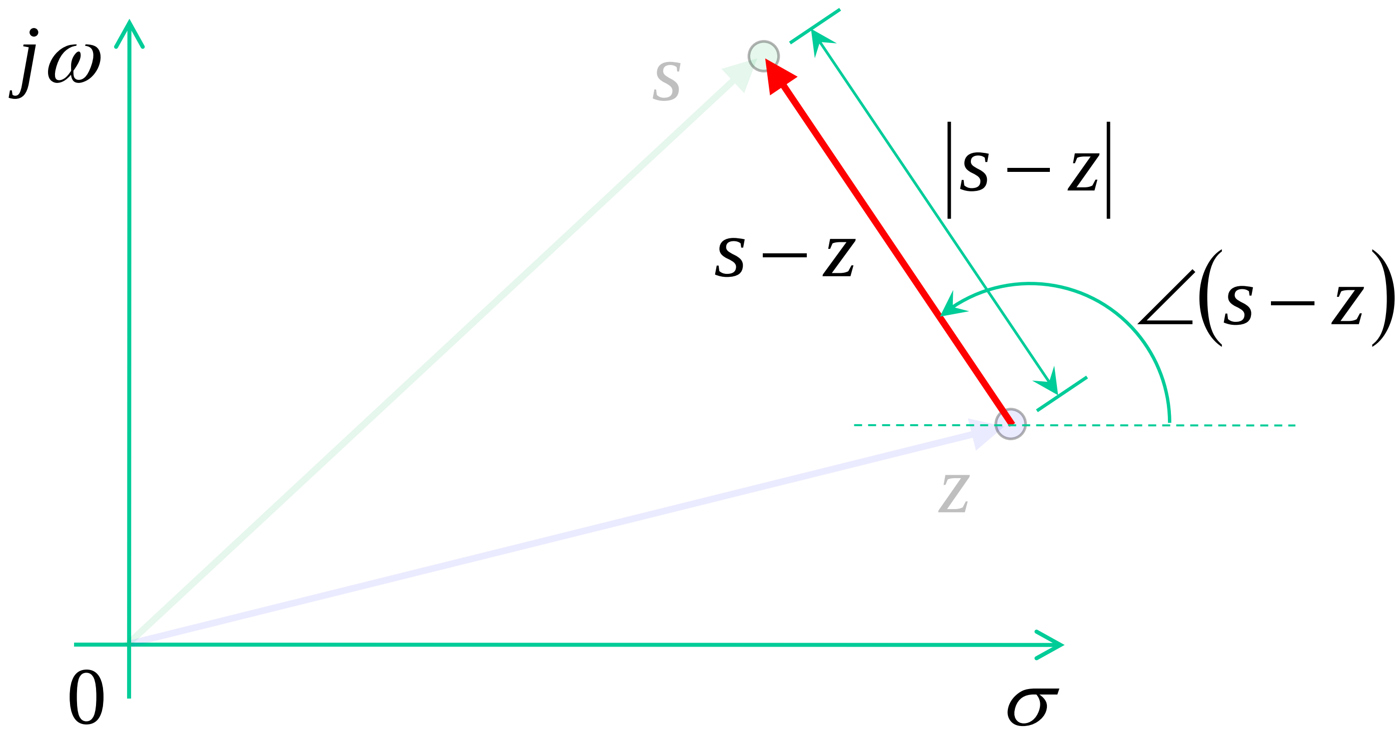
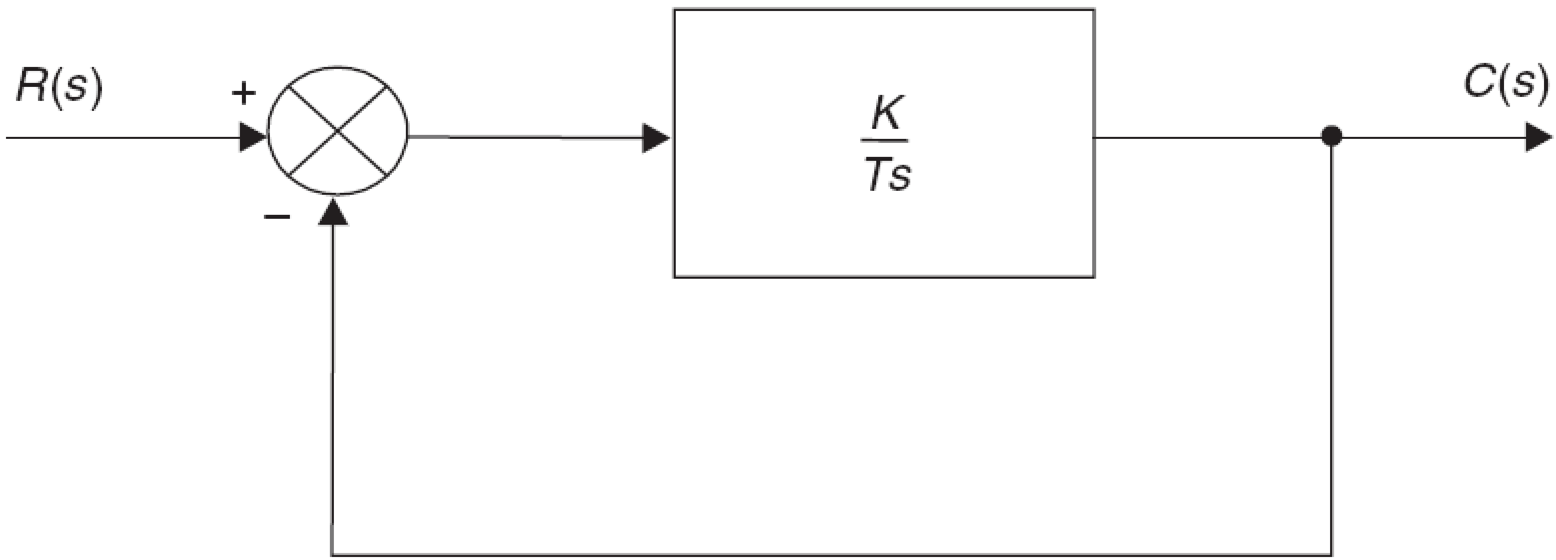


What is a Root Locus?

- Root Locus (of closed-loop poles)
  - Trajectory along which a (closed-loop) pole "moves" as a parameter, usually some open-loop gain K, of the system is varied.
  - Locus or curve describing the position of a closed-loop pole. This locus/curve is parameterised by the parameter K.
  - For <u>each</u> closed-loop pole there is a locus.

Introductory Example 1

- Block diagram



Introductory Example 1

- Open-loop
  - Open-loop transfer function: $G(s)H(s) = \dfrac{K}{Ts}$
  - Open-loop poles: $s = 0$
  - Open-loop zeros: none
  - Closed-loop poles (characteristic equation):

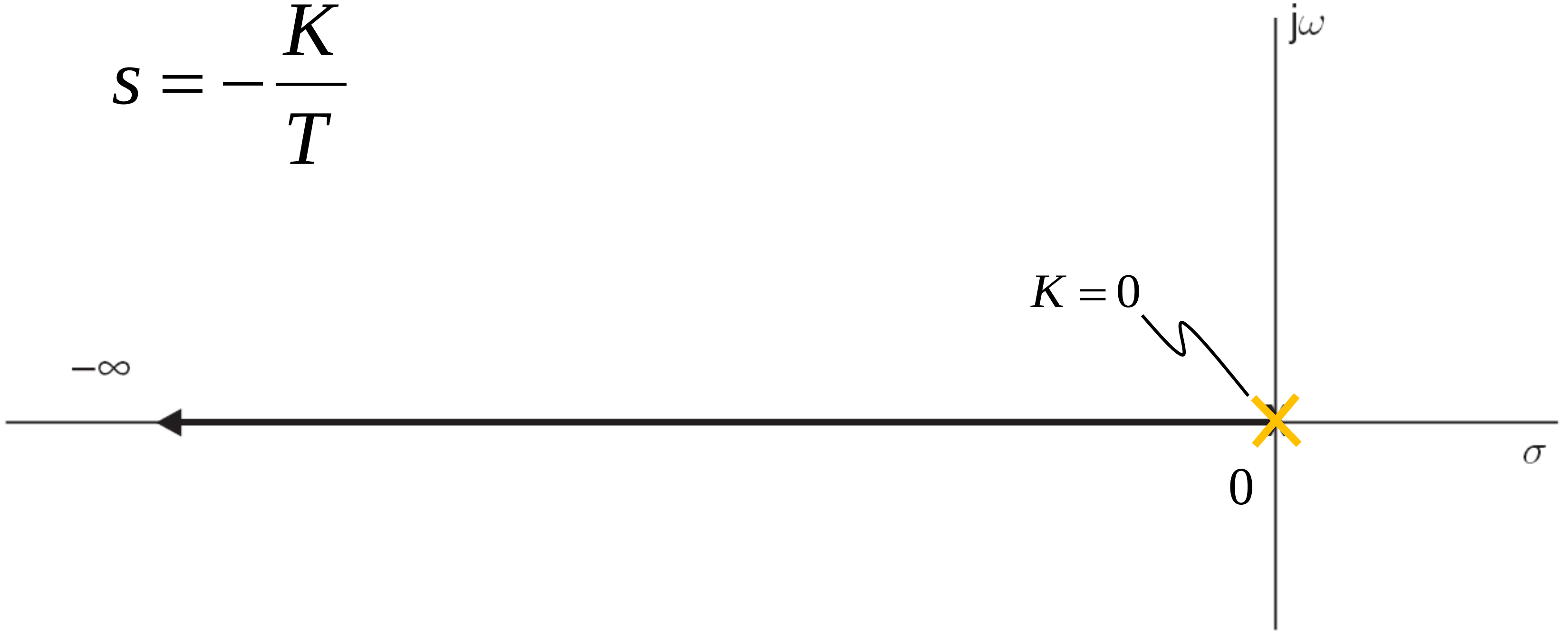
$$1 + G(s)H(s) = 0 \quad \Rightarrow \quad s = -\frac{K}{T}$$

  - Conclusion: Closed-loop pole position depends on K

Introductory Example 1

- Root Locus (of the closed-loop pole)
  - A single open-loop pole implies a single locus.

$$s = -\frac{K}{T}$$



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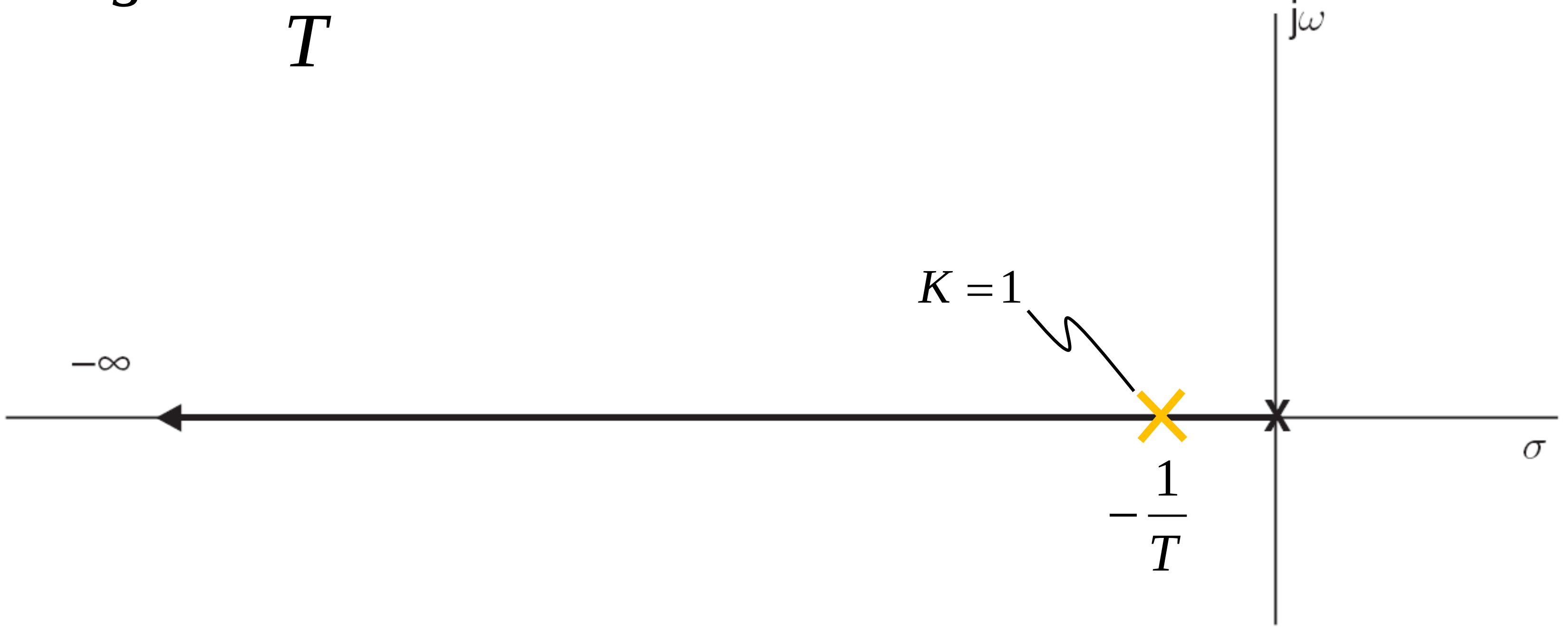

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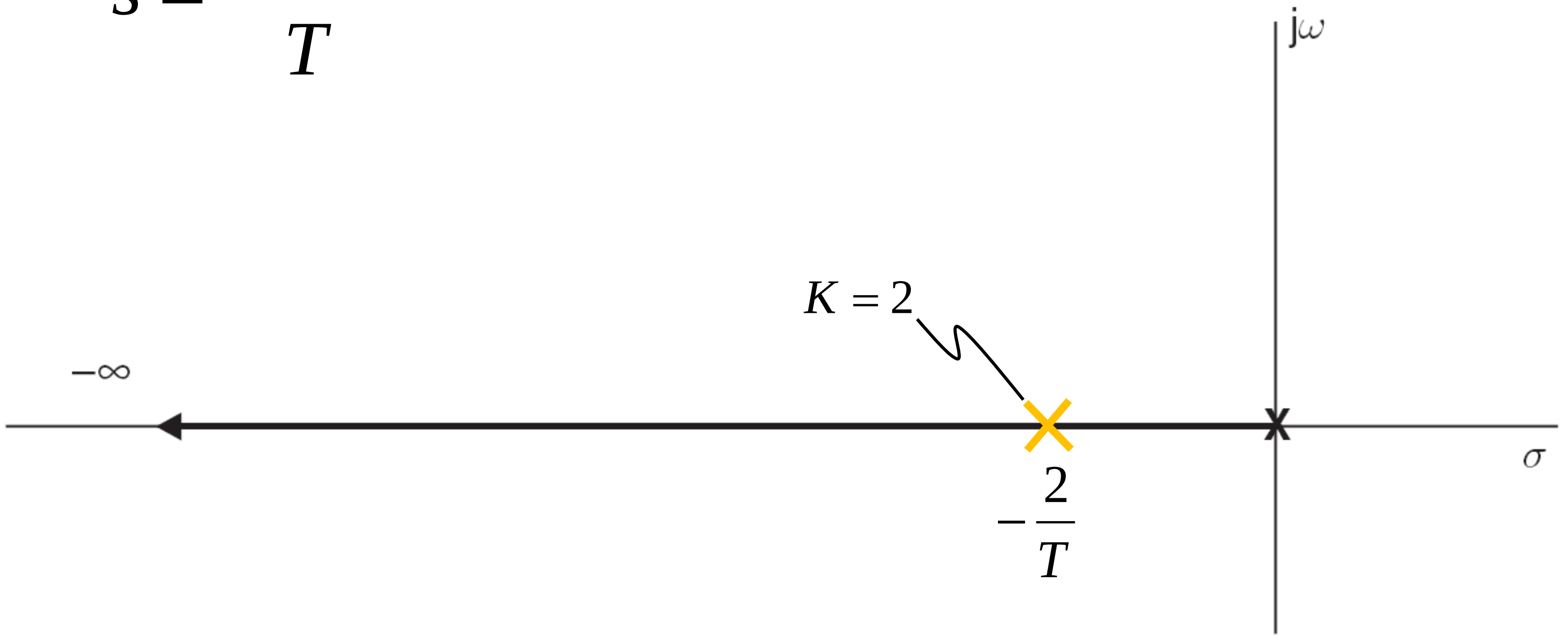

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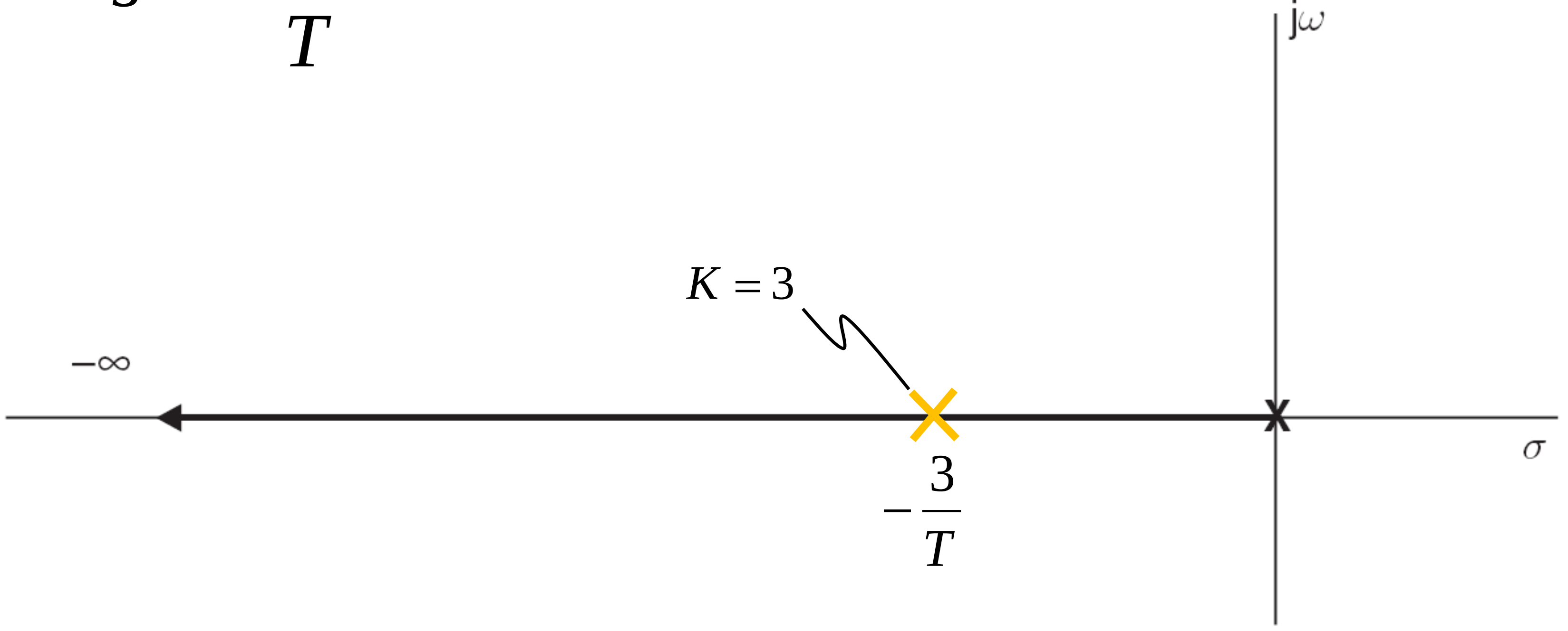

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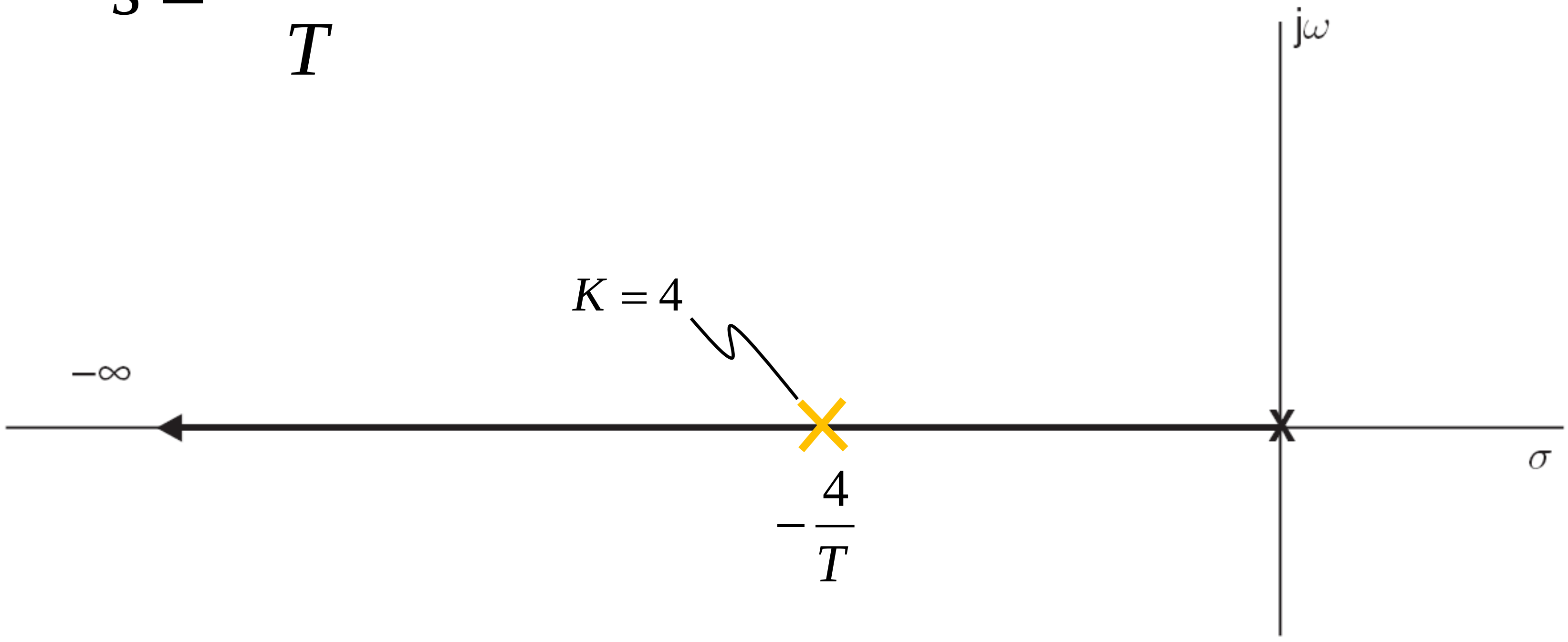

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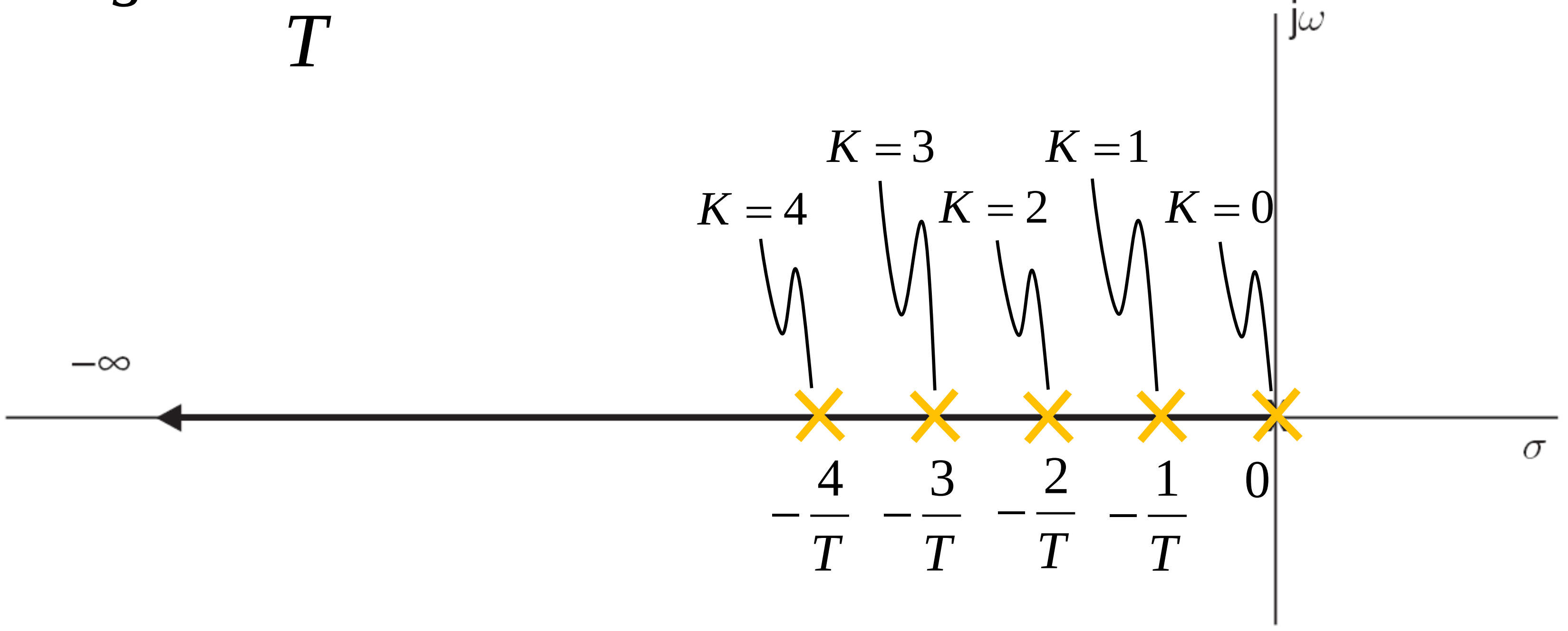
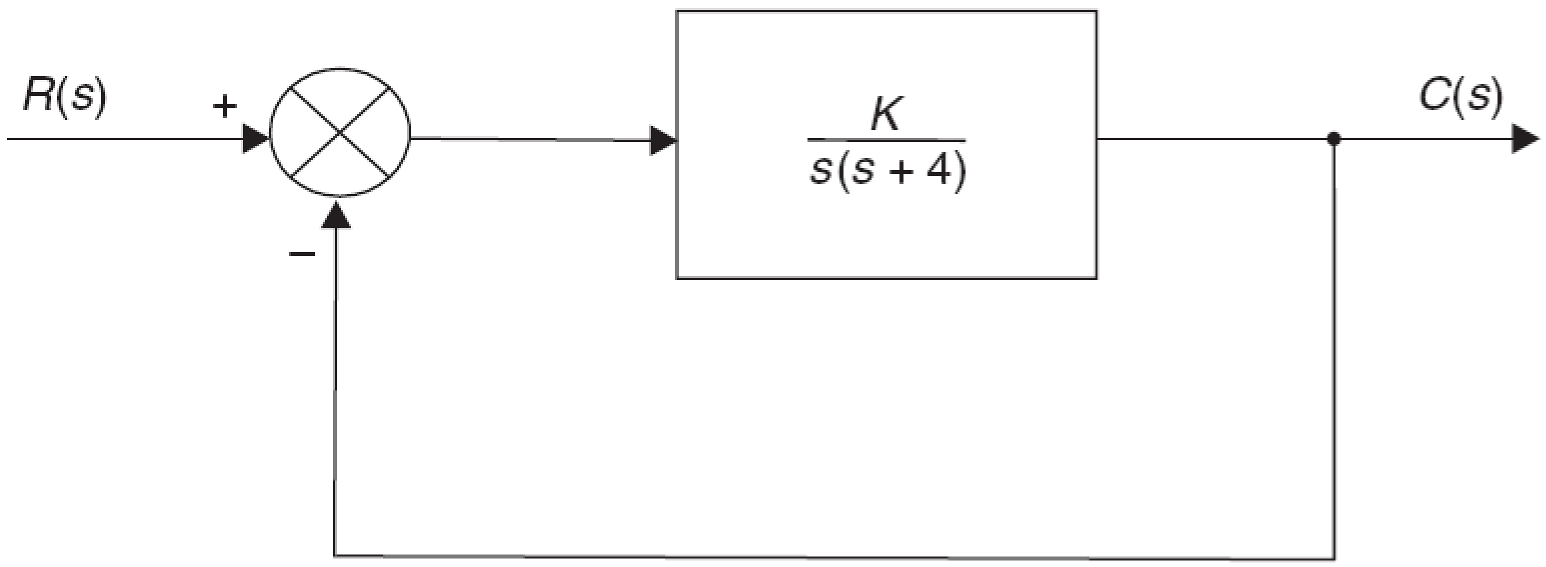


Introductory Example 2

- Block diagram



Introductory Example 2

- Open-loop
  - Open-loop transfer function: $G(s)H(s) = \dfrac{K}{s(s+4)}$
  - Open-loop poles: $s = 0, -4$
  - Open-loop zeros: none
  - Characteristic equation:

$$1 + G(s)H(s) = 0 \quad \Rightarrow \quad s^2 + 4s + K = 0$$

  - Conclusion: Poles' positions depend on K

Introductory Example 2

- Poles for different values of *K*

K	*Characteristic equation*	*Roots*
0	$s^2 + 4s = 0$	$s = 0, -4$
4	$s^2 + 4s + 4 = 0$	$s = -2 \pm j0$
8	$s^2 + 4s + 8 = 0$	$s = -2 \pm j2$
16	$s^2 + 4s + 16 = 0$	$s = -2 \pm j3.46$

$$s^2 + 4s + K = 0 \quad \Rightarrow \quad s = \frac{-4 \pm \sqrt{16 - 4K}}{2} = -2 \pm \sqrt{4 - K}$$

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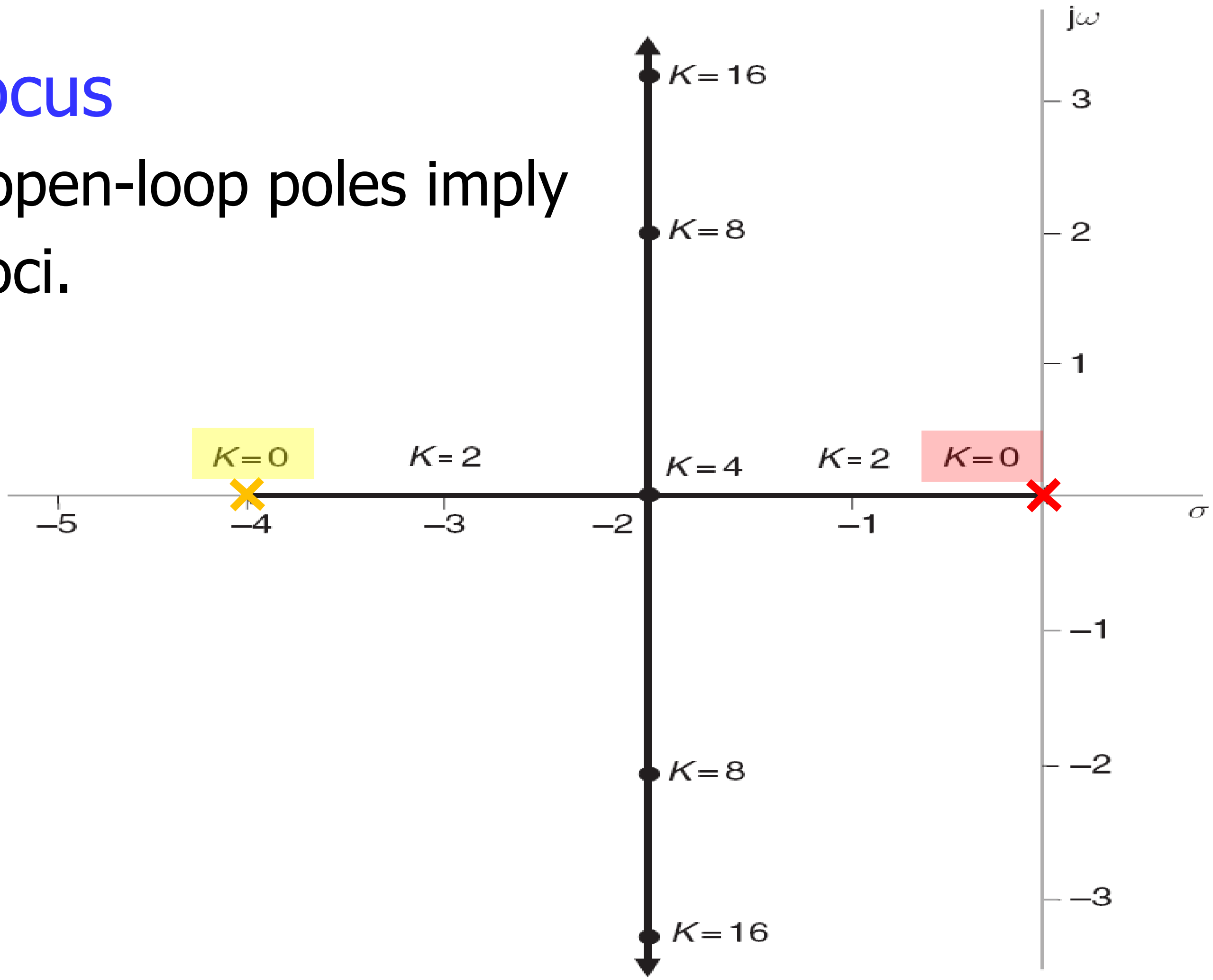
Introductory Example 2

- Root Locus
  - Two open-loop poles imply two loci.



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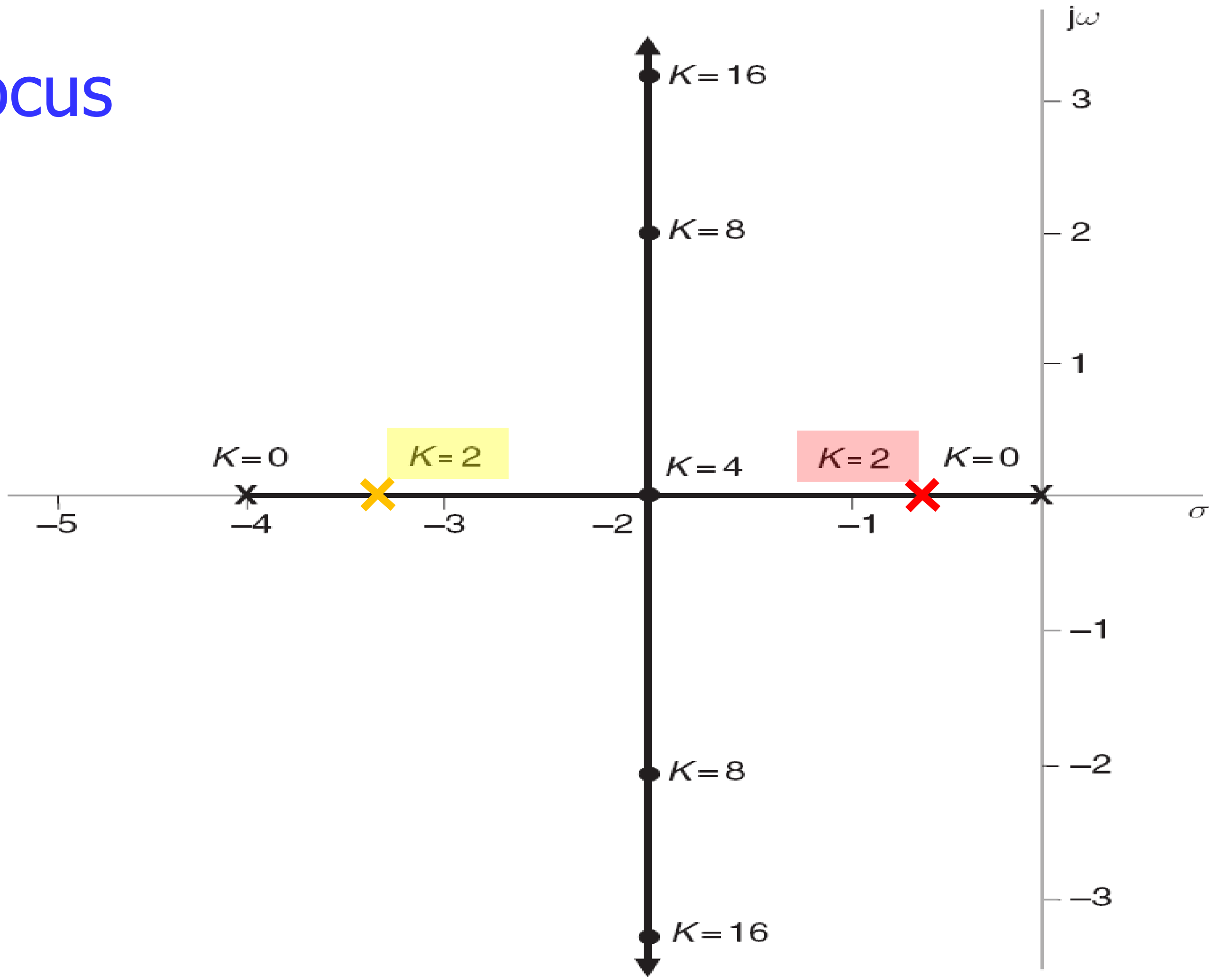

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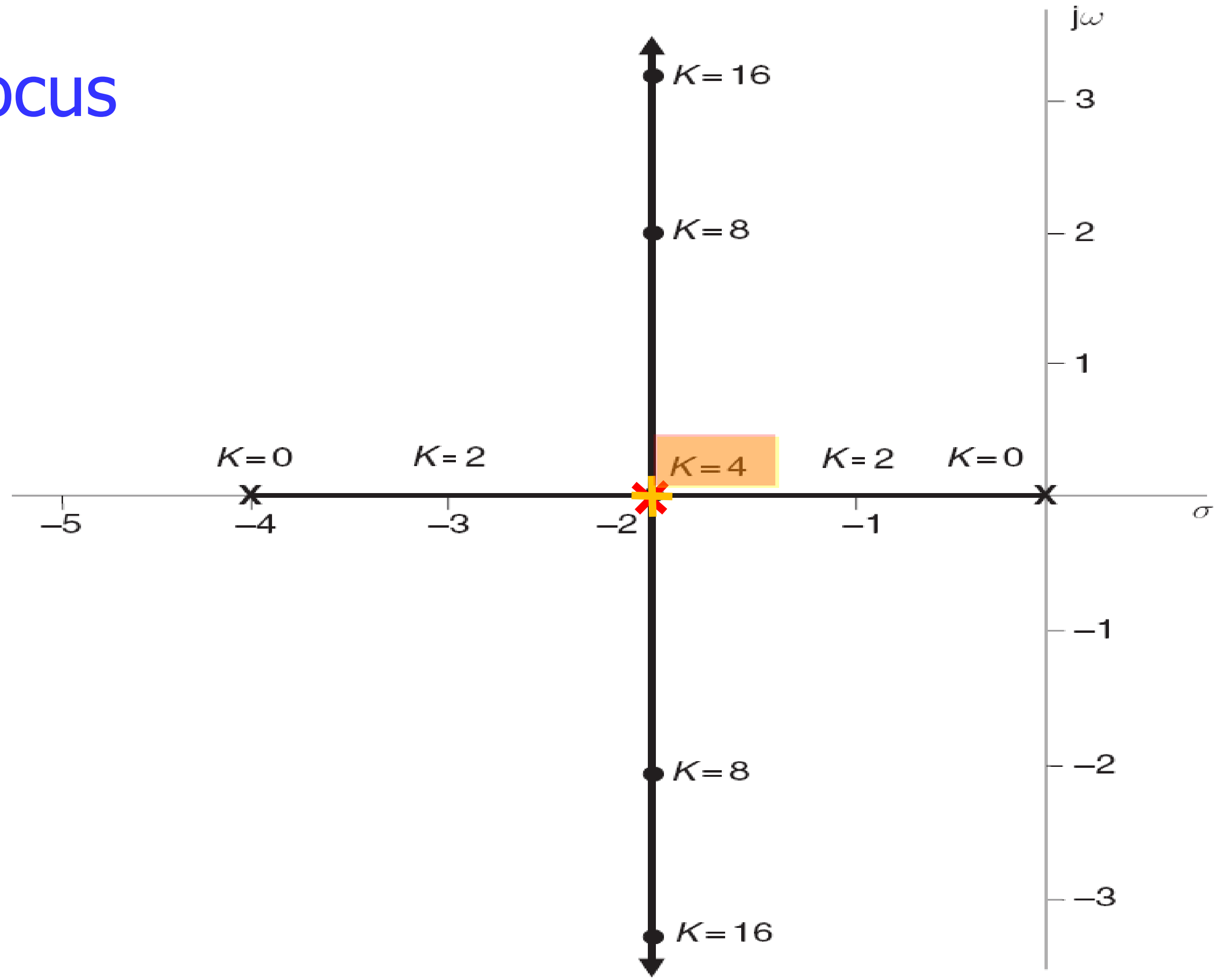

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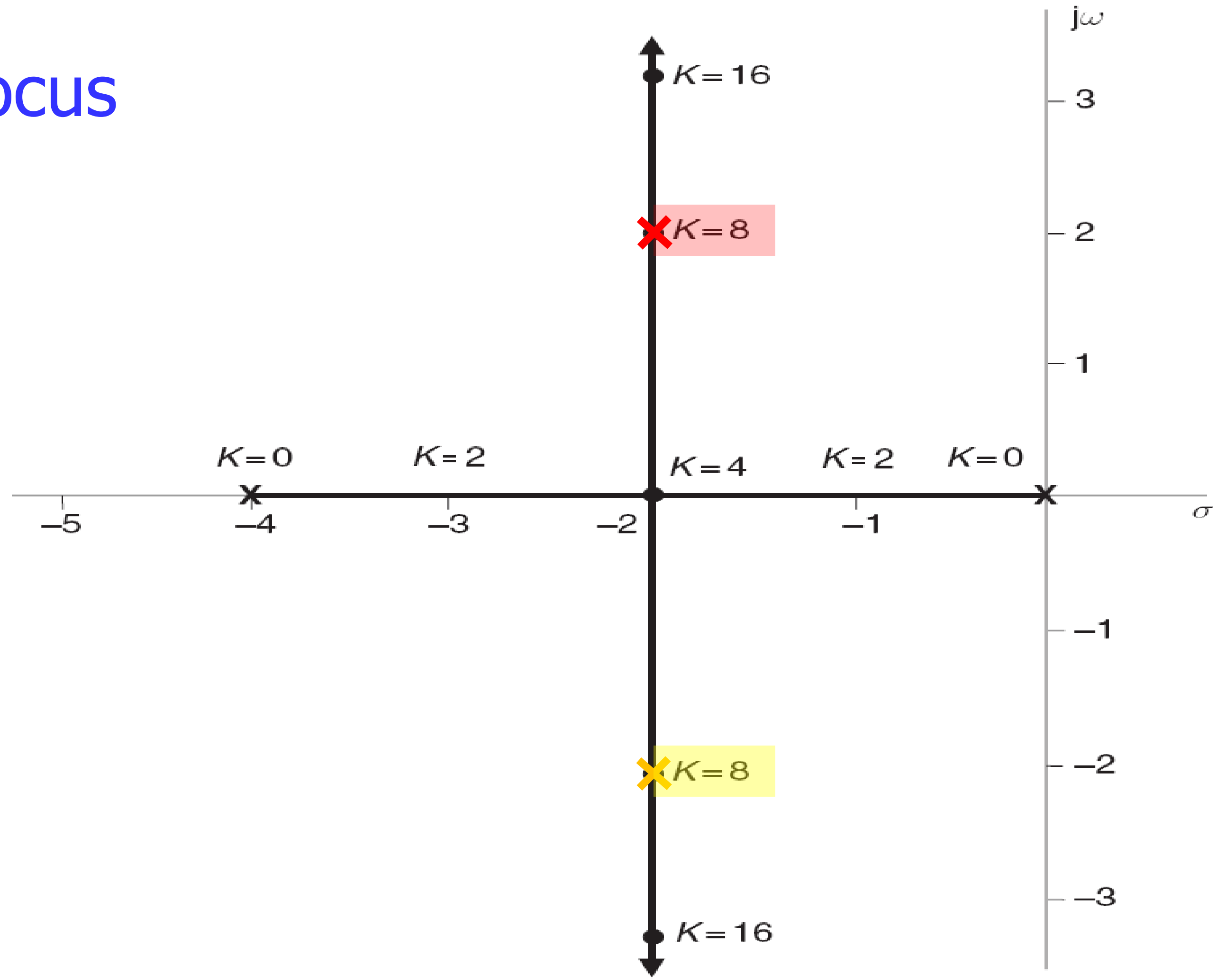

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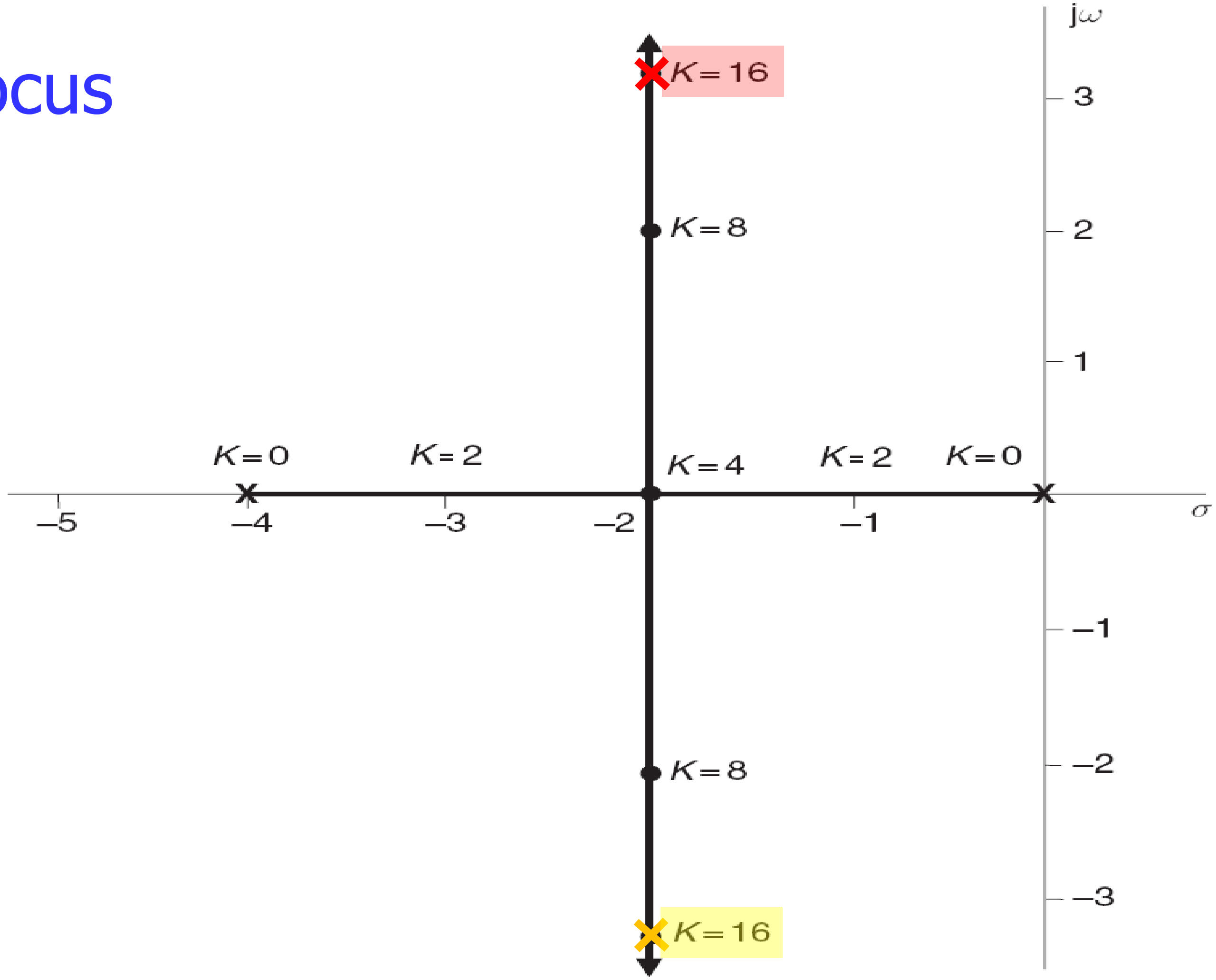

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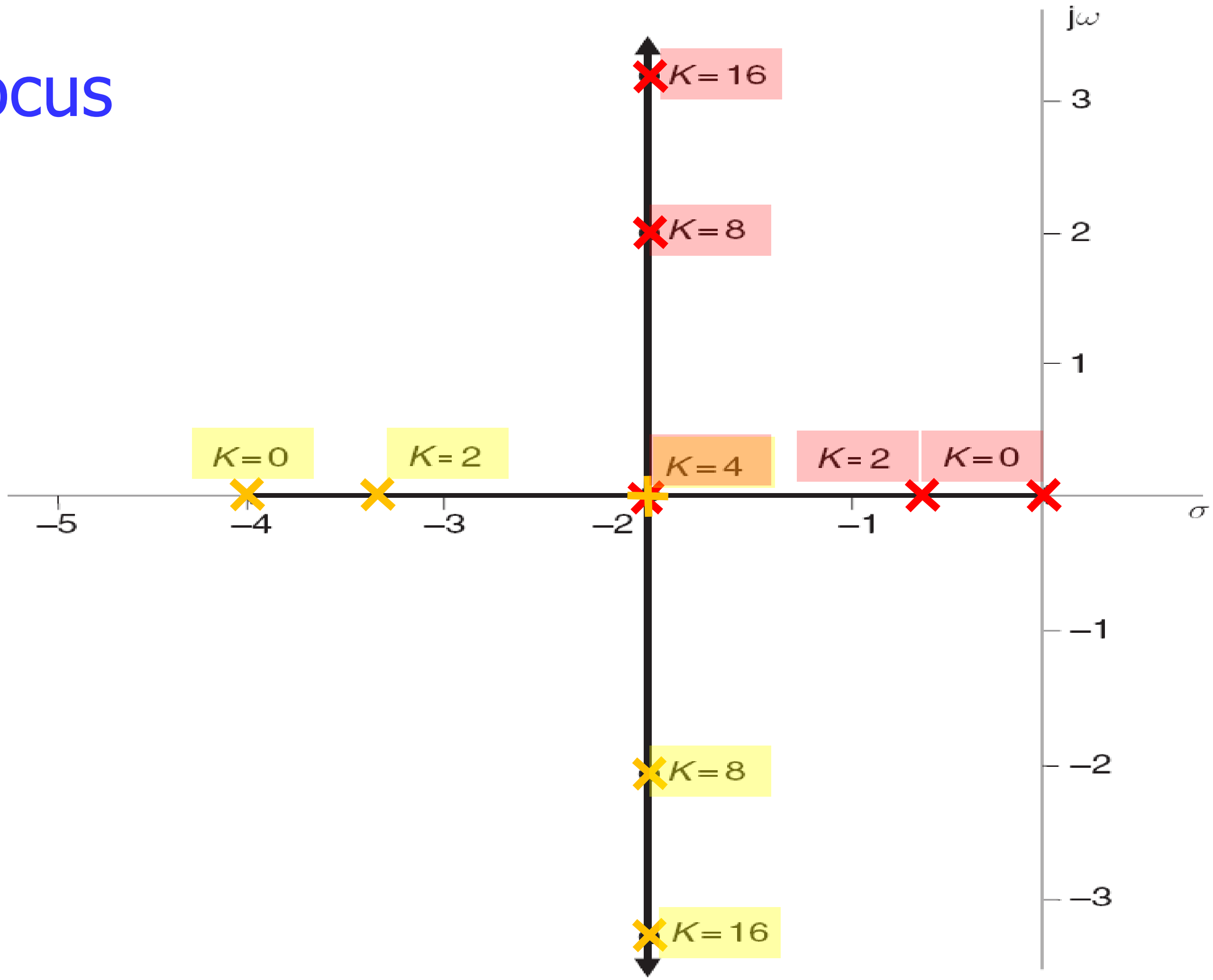


Root Locus

- Moral of the story …
  - Changing the gain within a closed loop containing a (non-trivial) plant causes the closed-loop pole positions to deviate from the (open-loop) plant's pole positions.

Root Locus

- The Root Locus Problem
  - Constructing the root locus for a given system means finding all ordered pairs $(\tilde{s}, \tilde{K})$ that satisfy the system's characteristic equation, that is,

$$1 + G(\tilde{s})H(\tilde{s})\Big|_{K=\tilde{K}} = 0.$$

  - The root locus rules (to come) ease this burden and enables us to swiftly sketch a root locus diagram with *minimal* effort.

Root Locus Rules

- Magnitude and Angle Criteria

$$1+G(s)H(s)=0 \quad \Rightarrow \quad G(s)H(s)=-1$$

  – This implies that the magnitude and phase satisfy

$$|G(s)H(s)|=1, \qquad \angle\big(G(s)H(s)\big)=180^\circ$$

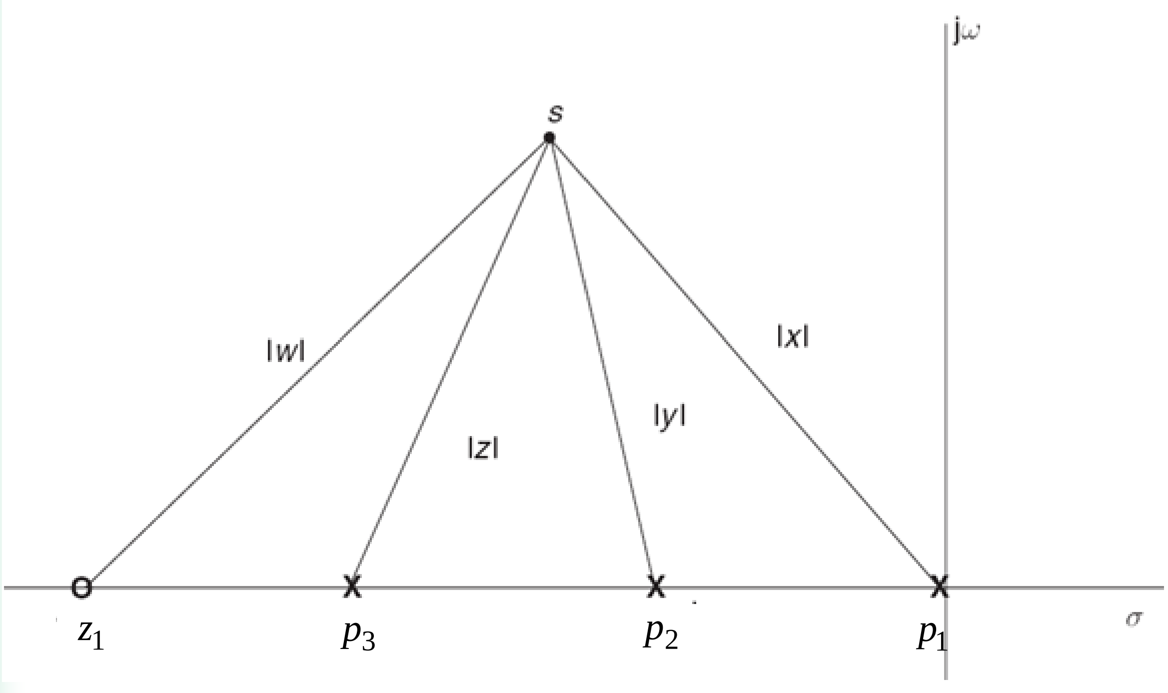
  – Assumption:

$$G(s)H(s)=K\frac{N(s)}{D(s)}, \qquad \begin{array}{c}\deg N(s)=m \\ \deg D(s)=n\end{array}$$

Root Locus Rules

- Magnitude Criterion

$$|G(s)H(s)| = 1 \quad \Rightarrow \quad |K| = \frac{|s-p_1|\times\cdots\times|s-p_n|}{|s-z_1|\times\cdots\times|s-z_m|}$$

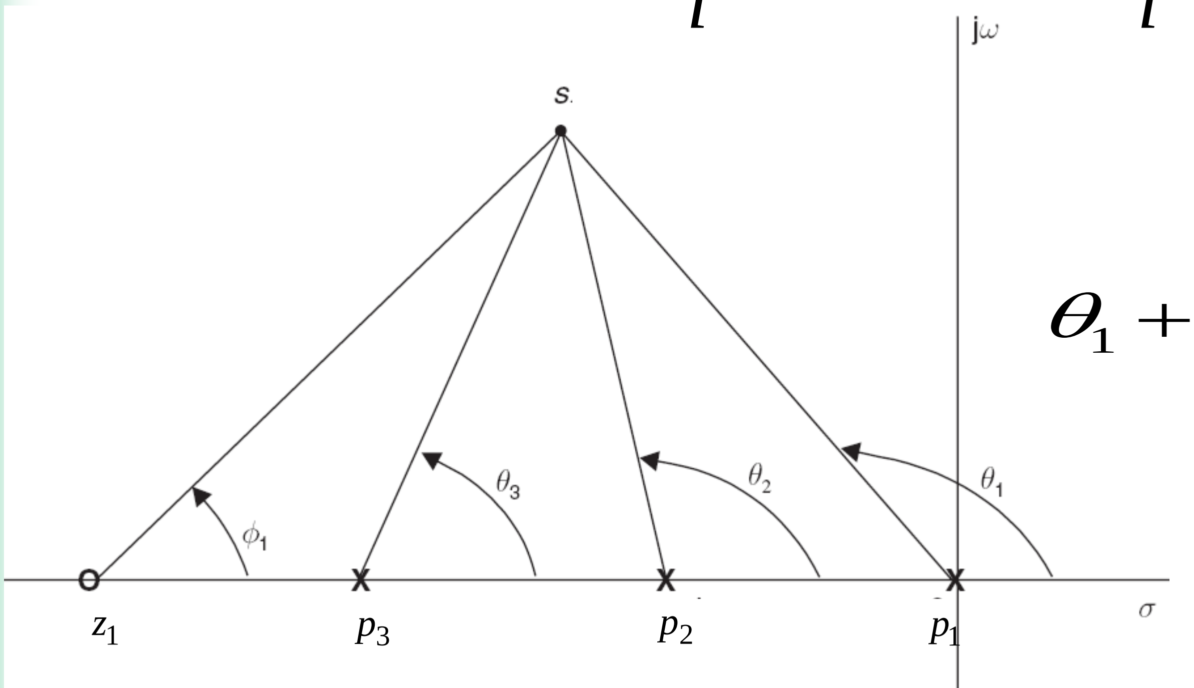


$$|K| = \frac{|x|\times|y|\times|z|}{|w|}$$

Root Locus Rules

- Angle Criterion

$$\angle\left(\frac{1}{G(s)H(s)}\right) = \sum_i \angle(s-p_i) - \sum_i \angle(s-z_i) = 180°$$



$$\theta_1 + \theta_2 + \theta_3 - \phi_1 = 180$$

Root Locus Rules

- Starting points $\left(K=0\right)$
  - Root loci start on open-loop poles
- Termination points $\left(K=\infty\right)$
  - Root loci terminate on open-loop zeros
- No. of distinct loci
  - Equal to the degree of the Characteristic Equation
- Symmetry of loci
  - Symmetric about the real axis

Root Locus Rules

- Asymptotes $\left(K \to \infty\right)$
  - For $K >> 1$ loci approach straight-line asymptotes with angles

$$\alpha_k = \frac{180^\circ + k360^\circ}{n-m}, \qquad k = 0, 1, \ldots, (n-m-1)$$

- Asymptotes' real axis intercept
  - All of the above asymptotes meet the real axis at

$$\sigma_a = \frac{\sum_{i=1}^{n} p_i - \sum_{j=1}^{m} z_j}{n-m}$$

Note: *Imaginary parts* of poles & zeros do not contribute to σ_a .

Root Locus Rules

- Root locus segments on the real axis
  - A point on the real axis is on a locus segment if there is an odd total number of poles & zeros to its right.
- Breakaway/break-in points
  - At points where loci break away from or into the real axis

$$\left.\frac{dK(s)}{ds}\right|_{s=\sigma_b} = 0 \qquad \text{with} \qquad K(s) = -\frac{D(s)}{N(s)}$$

- Angles of departure & arrival
  - Follow from the angle criterion in a small neighbour-hood of the point (i.e. pole or zero) of interest.

Tutorial Exercises & Homework

- Tutorial Exercises
  - Burns, Examples 5.14 and 5.15
  - D'Azzo & Houpis, Section 7-10, Example 2, pp. 235 – 240.
- Homework
  - Burns, Sections 5.3.1 and 5.3.3

Conclusion

- Root Locus Introductory Examples
- Root Locus Construction rules
- Burns, Sec 5.3.1, 5.3.3 (Self-study!)
  - Sec 5.3.3 suggests contours of constant ζ that can be extended to contours of constant ω_n, ω_d and $\zeta\omega_n$. (As discussed earlier this semester.)
- Tutorial Exercises & Homework

Next Attraction! – Miss It & You'll Miss Out!

- Applications of the Root Locus Technique (Burns, Chapter 5)

  ...

Thank you!

Any Questions?