CONTROL I

ELEN3016

Stability of Dynamical Systems

(Lecture 11)

Overview

- First Things First!
- Introductory Examples
- Review of 2^{nd}-Order Systems
- Routh-Hurwitz Stability Criterion
- Tutorial Exercises & Homework

- Next Attraction!

First Things First!

- Lab Related Matters
  - Student assistants:

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    John Ekoru – johnekoru@gmail.com

    Jarren Lange – jarrenlange@gmail.com
  - DLab booked: Mondays & Tuesdays, 14:00 – 17:00
  - Official Deadline for to be decided
  - Informal milestone dates?

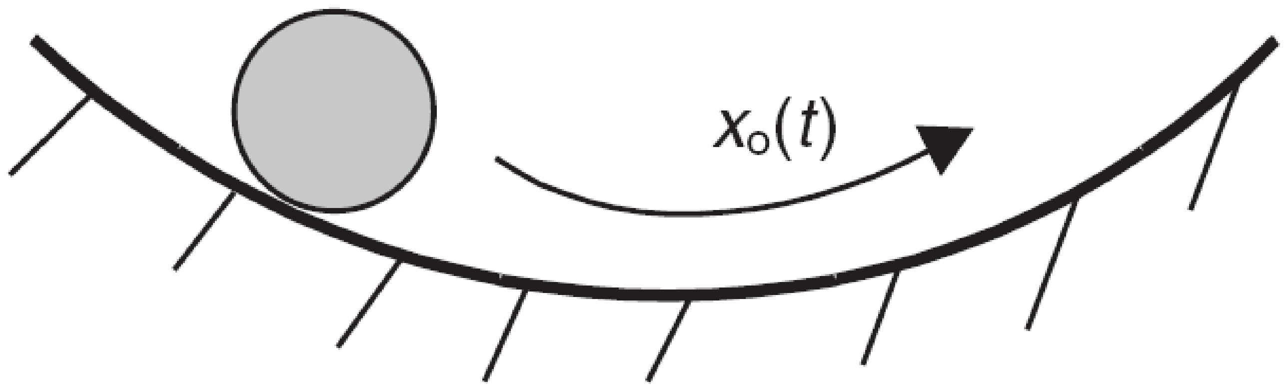
First Things First!

- Test
  - After the study break … ☺

    30th September, 08:00 – 10:00 in FNBBA.
  - Will cover Chapters 1 to 5 of Burns.
  - All problems discussed in class, self-study, homework, all tutorial problems and (small) variations on these themes are important for the Test.

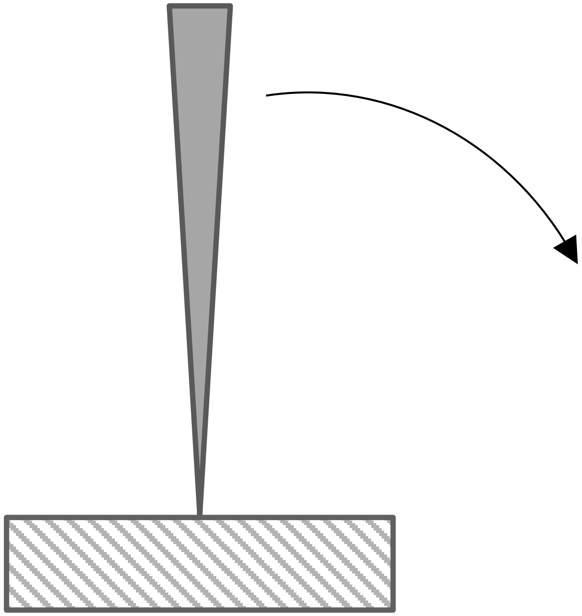
Introductory Example 1

- *Stable* time response



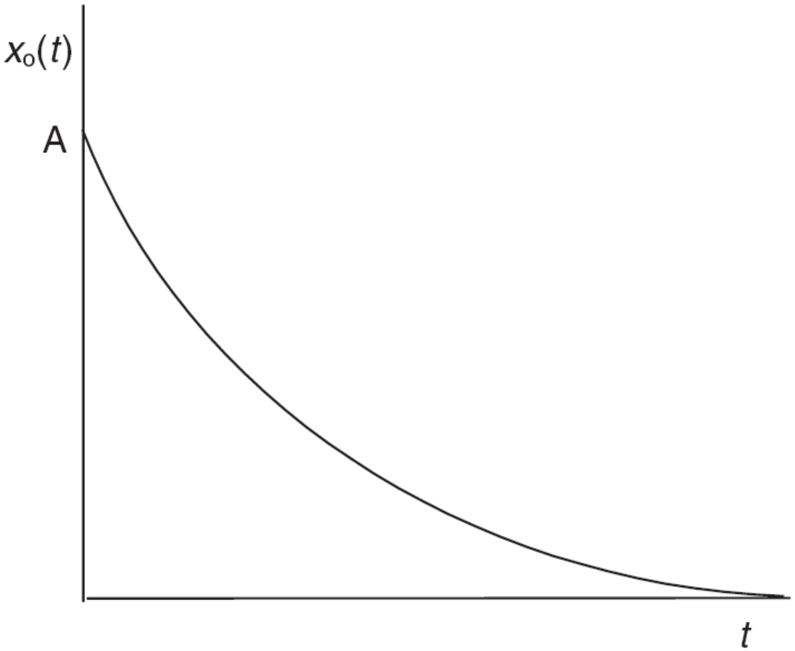
Introductory Example 2

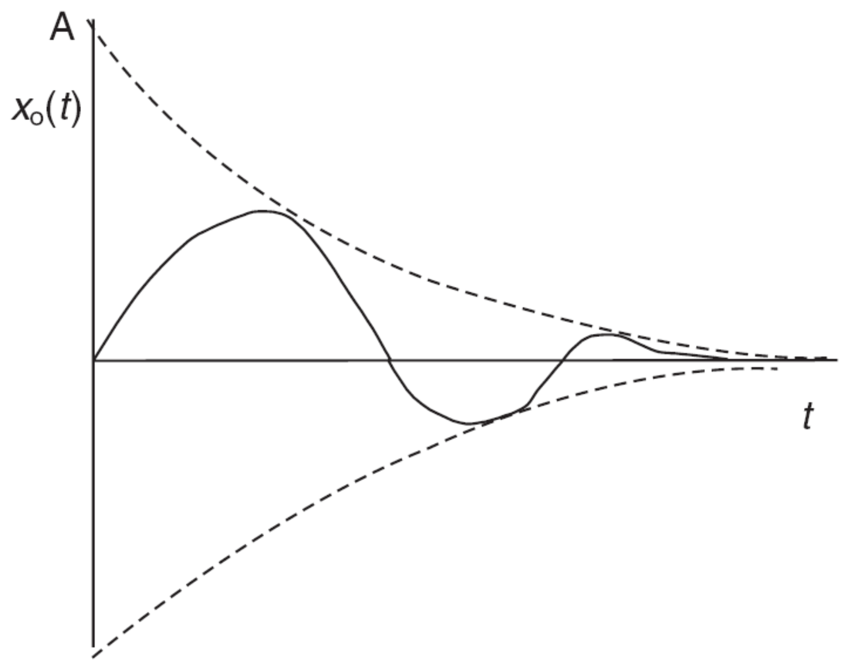
- Example of *unstable* time response



Types of Stable Time Response

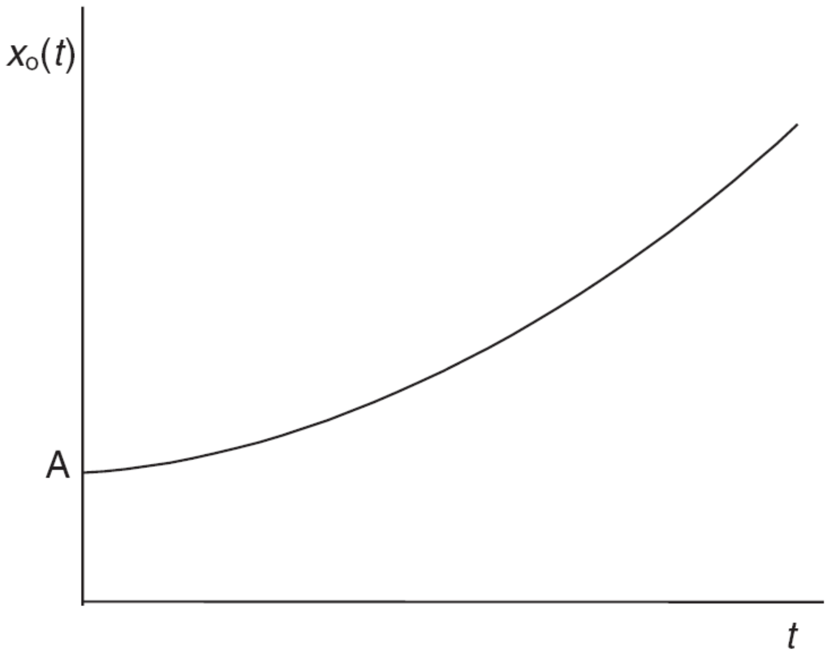
- Stable response: Non-oscillatory vs Oscillatory

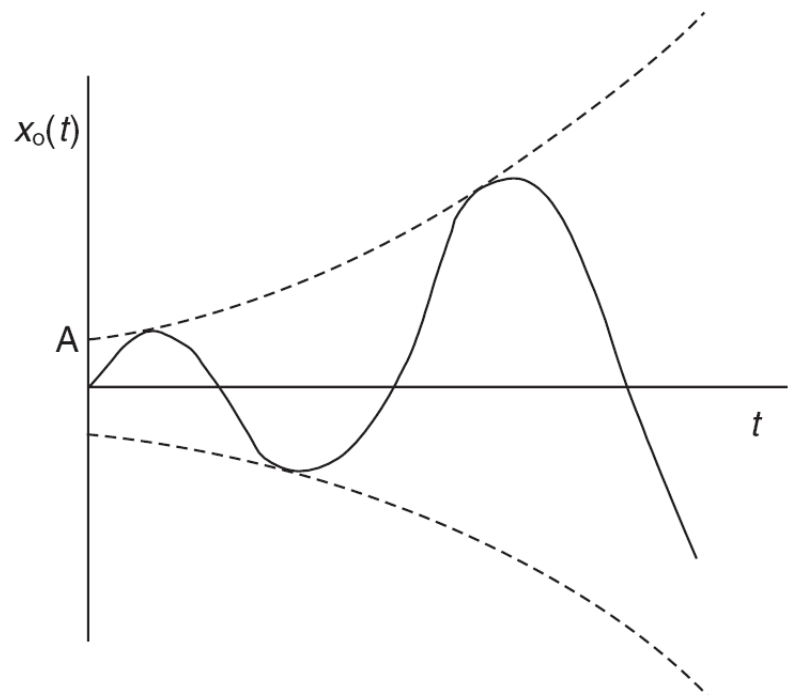




Types of Unstable Time Response

- Unstable response: Non-oscillatory vs Oscillatory





Characteristic Equation & Roots

- Characteristic equation of a 2nd-order system

$$as^2 + bs + c = 0 \tag{5.5}$$

- Roots

$$s_1,\ s_2 = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \tag{5.6}$$

$$\underbrace{a, b, c > 0}_{\text{That is, all of the same sign!}} \quad \Rightarrow \quad \begin{cases} \mathrm{Re}(s_1), \mathrm{Re}(s_2) < 0 \\ b > \sqrt{b^2 - 4ac} \end{cases}$$

Characteristic Equation & Roots

- Overdamped 2nd-order system $(b^2 - 4ac > 0)$

$$s_1 = -\sigma_1 \qquad s_2 = -\sigma_2 \tag{5.7}$$

- Critically damped 2nd-order system $(b^2 - 4ac = 0)$

$$s_1 = s_2 = -\sigma \tag{5.8}$$

- Underdamped 2nd-order system $(b^2 - 4ac < 0)$

$$s_1,\, s_2 = -\sigma \pm \mathrm{j}\omega \tag{5.9}$$

Characteristic Equation & Roots

- 2^{nd}-order system with $a, c > 0,\ \ b < 0$ implies
  - a. $-4ac < 0 \Rightarrow\ b^2 - 4ac < b^2$ implying either
    - i. complex conjugate pole pair if $b^2 - 4ac < 0$ **<u>or</u>**
    - ii. two real poles of the same sign if $0 < b^2 - 4ac$.
  - b. $-\dfrac{b}{2a} > 0$ i.e. at least one unstable pole.

Combining a.) & b.) we conclude that both poles (real or complex) are in the RHP giving an <u>unstable</u> system.

Characteristic Equation & Roots

- 2^{nd}-order system with $b, c > 0, \quad a < 0$ implies

  a. $-4ac > 0 \Rightarrow b^2 - 4ac > b^2$ from which we have two real poles of opposite signs since $0 < b < \sqrt{b^2 - 4ac}$.

  b. $-\frac{b}{2a} > 0$ i.e. at least one unstable pole.

Combining a.) & b.) ensures a single unstable pole in the RHP giving an <u>unstable</u> system.

Characteristic Equation & Roots

- 2nd-order system with $a, b > 0,\ \ c < 0$ implies

  a. $-4ac > 0 \Rightarrow b^2 - 4ac > b^2$ from which we have two real poles of opposite signs since $0 < b < \sqrt{b^2 - 4ac}$.

  b. $-\frac{b}{2a} < 0$ i.e. at least one stable pole.

Combining a.) & b.) ensures a single unstable pole in the RHP giving an <u>unstable</u> system.

Stability of a LTI System

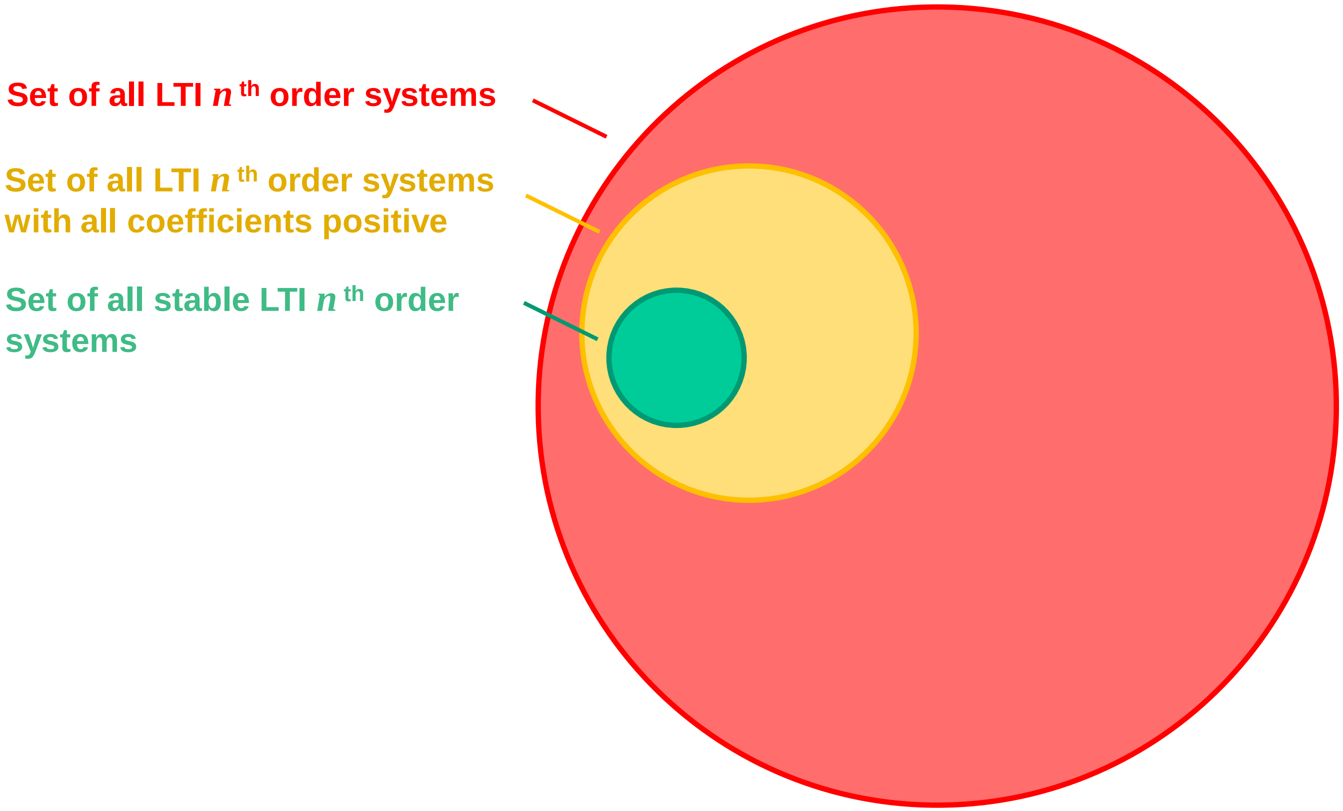
- Characteristic equation of an n th-order system

$$a_n s^n + a_{n-1} s^{n-1} + \cdots + a_1 s + a_0 = 0$$

- Necessary but not sufficient condition for a system to be stable:
  - The coefficients of the characteristic equation must all have the same sign and that none are zero.

Stability of a LTI System

- Set-theoretic depiction of all LTI n th order systems



Comment on the closedness and openness of these sets.

Routh-Hurwitz Criterion

- Characteristic equation of an n th-order system

$$a_n s^n + a_{n-1} s^{n-1} + \cdots + a_1 s + a_0 = 0 \quad (5.11)$$

- Necessary and sufficient condition for a system to be stable:
  - All the *Hurwitz determinants* of the characteristic polynomial are positive or, equivalently,
  - All coefficients in the first column of the *Routh array* have the same sign.

Routh-Hurwitz Criterion

- Routh array

s^0	p_1			
s^1	q_1			
$\vdots$	$\vdots$			
s^{n-3}	c_1	c_2	c_3	$\cdots$
s^{n-2}	b_1	b_2	b_3	$\cdots$
s^{n-1}	a_{n-1}	a_{n-3}	a_{n-5}	$\cdots$
s^n	a_n	a_{n-2}	a_{n-4}	$\cdots$

Routh-Hurwitz Criterion

- Routh array

$$
\begin{array}{c|cccc}
s^0 & p_1 & & & \\
s^1 & q_1 & & & \\
\vdots & \vdots & & & \\
s^{n-3} & c_1 & c_2 & c_3 & \cdots \\
s^{n-2} & b_1 & b_2 & b_3 & \cdots \\
s^{n-1} & a_{n-1} & a_{n-3} & a_{n-5} & \cdots \\
s^n & a_n & a_{n-2} & a_{n-4} & \cdots \\
\hline
\end{array}
$$

Routh-Hurwitz Criterion

- Routh array

$$
\begin{array}{l|llll}
s^0 & p_1 & & & \\
s^1 & q_1 & & & \\
\vdots & \vdots & & & \\
s^{n-3} & c_1 & c_2 & c_3 & \cdots \\
s^{n-2} & b_1 & b_2 & b_3 & \cdots \\
s^{n-1} & a_{n-1} & a_{n-3} & a_{n-5} & \cdots \\
s^n & a_n & a_{n-2} & a_{n-4} & \cdots \\
\hline
\end{array}
$$

Routh-Hurwitz Criterion

- Developing the Routh array's third row:

$$b_1 = \frac{1}{a_{n-1}} \begin{vmatrix} a_{n-1} & a_{n-3} \\ a_n & a_{n-2} \end{vmatrix}$$

$$b_2 = \frac{1}{a_{n-1}} \begin{vmatrix} a_{n-1} & a_{n-5} \\ a_n & a_{n-4} \end{vmatrix}$$

etc.

  – Continue until the first zero appears. Then move to next row.

Routh-Hurwitz Criterion

- Routh array

$$b_1 = \frac{1}{a_{n-1}} \begin{vmatrix} a_{n-1} & a_{n-3} \\ a_n & a_{n-2} \end{vmatrix}$$

$$\begin{array}{c|cccc}
s^0 & p_1 & & & \\
s^1 & q_1 & & & \\
\vdots & \vdots & & & \\
s^{n-3} & c_1 & c_2 & c_3 & \cdots \\
s^{n-2} & b_1 & b_2 & b_3 & \cdots \\
s^{n-1} & a_{n-1} & a_{n-3} & a_{n-5} & \cdots \\
s^n & a_n & a_{n-2} & a_{n-4} & \cdots \\
\hline
\end{array}$$

Routh-Hurwitz Criterion

- Routh array

$$\boxed{b_2 = \frac{1}{a_{n-1}} \begin{vmatrix} a_{n-1} & a_{n-5} \\ a_n & a_{n-4} \end{vmatrix}}$$

$$\begin{array}{c|llll}
s^0 & p_1 & & & \\
s^1 & q_1 & & & \\
\vdots & \vdots & & & \\
s^{n-3} & c_1 & c_2 & c_3 & \cdots \\
s^{n-2} & b_1 & b_2 & b_3 & \cdots \\
s^{n-1} & a_{n-1} & a_{n-3} & a_{n-5} & \cdots \\
s^n & a_n & a_{n-2} & a_{n-4} & \cdots \\
\hline
\end{array}$$

Routh-Hurwitz Criterion

- Developing the Routh array's fourth row:

$$c_1 = \frac{1}{b_1} \begin{vmatrix} b_1 & b_2 \\ a_{n-1} & a_{n-3} \end{vmatrix}$$

$$c_2 = \frac{1}{b_1} \begin{vmatrix} b_1 & b_3 \\ a_{n-1} & a_{n-5} \end{vmatrix}$$

etc.

  - Continue until the first zero appears. Then move to next row. Terminate with a complete row of zeros.

Routh-Hurwitz Criterion

- Routh array

$$\boxed{c_1 = \frac{1}{b_1}\begin{vmatrix} b_1 & b_2 \\ a_{n-1} & a_{n-3} \end{vmatrix}}$$

s^0	p_1			
s^1	q_1			
$\vdots$	$\vdots$			
s^{n-3}	c_1	c_2	c_3	$\cdots$
s^{n-2}	b_1	b_2	b_3	$\cdots$
s^{n-1}	a_{n-1}	a_{n-3}	a_{n-5}	$\cdots$
s^n	a_n	a_{n-2}	a_{n-4}	$\cdots$

Routh-Hurwitz Criterion

- Routh array

$$\boxed{c_2 = \frac{1}{b_1}\begin{vmatrix} b_1 & b_3 \\ a_{n-1} & a_{n-5} \end{vmatrix}}$$

$$\begin{array}{c|cccc}
s^0 & p_1 & & & \\
s^1 & q_1 & & & \\
\vdots & \vdots & & & \\
s^{n-3} & c_1 & c_2 & c_3 & \cdots \\
s^{n-2} & b_1 & b_2 & b_3 & \cdots \\
s^{n-1} & a_{n-1} & a_{n-3} & a_{n-5} & \cdots \\
s^n & a_n & a_{n-2} & a_{n-4} & \cdots \\
\hline
\end{array}$$

Routh-Hurwitz Criterion

- Example 5.1

$$s^4 + 2s^3 + s^2 + 4s + 2 = 0$$

Routh array:

$$\begin{array}{c|ccc}
s^0 & 2 & & \\
s^1 & 8 & & \\
s^2 & -1 & 2 & \\
s^3 & 2 & 4 & \\
s^4 & 1 & 1 & 2 \\
\hline
\end{array}$$

Routh-Hurwitz Criterion

- Example 5.1 (cont'd)
  - Below shown the zeros that guide the calculations.

Routh array:

$$
\begin{array}{c|cccc}
 & 0 & 0 & 0 & 0 \\
s^0 & 2 & 0 & & \\
s^1 & 8 & 0 & & \\
s^2 & -1 & 2 & 0 & \\
s^3 & 2 & 4 & 0 & \\
s^4 & 1 & 1 & 2 & 0 \\
\hline
\end{array}
$$

Routh-Hurwitz Criterion

- Example 5.1 (cont'd)
  - Below shown the zeros that guide the calculations.

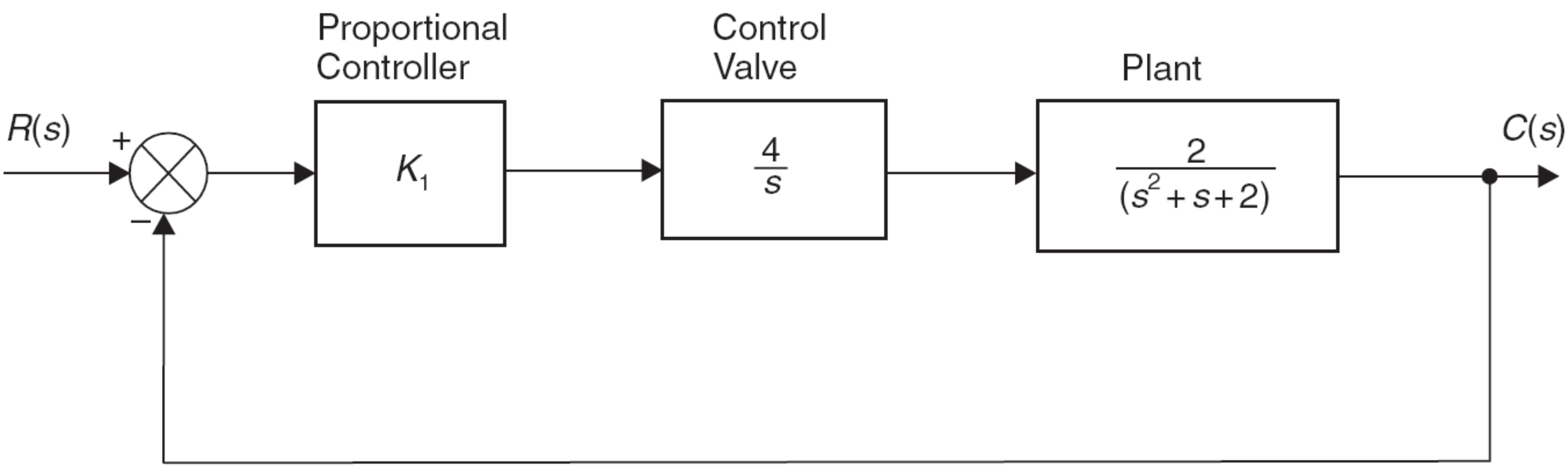
Routh array:

- 2 sign changes
- 2 poles in RHP

	0	0	0	0
s^0	2	0		
s^1	8	0		
s^2	-1	2	0	
s^3	2	4	0	
s^4	1	1	2	0

Routh-Hurwitz Criterion

- Example 5.2



Find the minimum proportional gain for which the system is marginally stable (i.e. “only just unstable”).

Routh-Hurwitz Criterion

- Example 5.2 (cont'd)

Characteristic equation: $(K = 8K_1)$

$$1 + G(s)H(s) = 1 + \frac{K}{s(s^2 + s + 2)} = 0$$

$$s^3 + s^2 + 2s + K = 0 \qquad (5.30)$$

Routh-Hurwitz Criterion

- Example 5.2 (cont'd)

Routh array:

$$
\begin{array}{c|cc}
s^3 & 1 & 2 \\
s^2 & 1 & K \\
s^1 & (2-K) & \\
s^0 & K &
\end{array}
$$

Routh-Hurwitz Criterion

- Example 5.2 (cont'd)

Routh array: Suppose $K = \varepsilon, \quad 0 < \varepsilon \ll 1$

$$
\begin{array}{c|cc}
s^3 & 1 & 2 \\
s^2 & 1 & \varepsilon \\
s^1 & 2 & \\
s^0 & \varepsilon &
\end{array}
$$

- 0 sign changes
- 0 poles in RHP

Routh-Hurwitz Criterion

- Example 5.2 (cont'd)

Routh array: Suppose $K = 2 - \varepsilon, \quad 0 < \varepsilon \ll 1$

- 0 sign changes
- 0 poles in RHP

$$
\begin{array}{c|cc}
s^3 & 1 & 2 \\
s^2 & 1 & 2 \\
s^1 & \varepsilon & \\
s^0 & 2 &
\end{array}
$$

Routh-Hurwitz Criterion

- Example 5.2 (cont'd)

Routh array: Suppose $K = 2 + \varepsilon, \quad 0 < \varepsilon \ll 1$

- 2 sign changes
- 2 poles in RHP

$$
\begin{array}{c|cc}
s^3 & 1 & 2 \\
s^2 & 1 & 2 \\
s^1 & -\varepsilon & \\
s^0 & 2 &
\end{array}
$$

Routh-Hurwitz Criterion

- Example 5.2 (cont'd)
  - Stability requires no sign changes in 1st column.

Routh array:

$$\begin{array}{c|cc} s^0 & K & \\ s^1 & (2-K) & \\ s^2 & 1 & K \\ s^3 & 1 & 2 \\ \hline \end{array} \longrightarrow \quad K \geq 2$$

$$K_1 = 0.25$$

Routh-Hurwitz Criterion

- Example 5.2 (cont'd)

  Study the remaining part of Example 5.2 as well as Section 5.2.2 on *special cases* of the Routh array.

Tutorial Exercises & Homework

- Tutorial Exercises
  - Burns, Examples 5.12 and 5.13
- Homework
  - Burns, Example 5.2 and Sec. 5.2.2

Conclusion

- First Things First!
- Introductory Examples
- Review of 2nd-Order Systems' Stability
- Routh-Hurwitz Stability Criterion
- Burns, Sec 5.2.2 (Self-study!)
- Tutorial Exercises & Homework

Next Attraction! – Miss It & You'll Miss Out!

- The Root Locus Technique (Burns, Chapter 5)

  ...

Thank you!

Any Questions?