Frequency Domain Analysis of Signals and Systems

ELEN 3024 - Communication Fundamentals

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July 15, 2013

Amplitude Modulation

Proakis and Salehi, "Communication Systems Engineering" (2nd Ed.), Chapter 2

Overview

Power and Energy

2.3 Power and Energy

Energy and power of signal representatives of energy or power delivered by the signal when signal is interpreted as voltage or current feeding 1 ohm resistor

Energy content of a signal:

$$\mathcal{E}_x = \int_{-\infty}^{\infty} |x(t)|^2 \mathrm{dt}$$

Power content of signal

$$P_x = \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} |x(t)|^2 \mathrm{dt}$$

2.3 Power and Energy

A signal is energy-type if $\mathcal{E}_x < \infty$

A signal is power-type if $0 < P_x < \infty$

Signal cannot be both power- and energy-type

Energy-type signals $\rightarrow P_x = 0$

Power-type signals $\rightarrow \mathcal{E}_x = \infty$

Signal can be neither energy-type nor power-type

2.3 Power and Energy

Most of the signals of interest either energy-type or power-type

All periodic signals are power-type:

$$P_x = \frac{1}{T_0}\int_{\alpha}^{\alpha+T_0} |x(t)|^2 \mathrm{dt}$$

2.3.1. Energy-Type Signals

Energy-type signal $x(t)$, autocorrelation function $R_x(\tau)$

$$\begin{aligned} R_x(\tau) &= x(\tau) \star x^*(-\tau) \\ &= \int_{-\infty}^{\infty} x(t)x^*(t-\tau)\mathrm{dt} \\ &= \int_{-\infty}^{\infty} x(t+\tau)x^*(t)\mathrm{dt} \end{aligned}$$

By setting $\tau = 0$, we obtain energy content:

$$\begin{aligned} \mathcal{E}_x &= \int_{-\infty}^{\infty} |x(t)|^2\mathrm{dt} \\ &= R_x(0) \end{aligned}$$

Fourier transform of $R_x(\tau) \Leftrightarrow |X(f)|^2$

$$\begin{aligned} \therefore \mathcal{E}_x &= \int_{-\infty}^{\infty} |x(t)|^2\mathrm{dt} \\ &= \int_{-\infty}^{\infty} |X(f)|^2\mathrm{df} \end{aligned}$$

Thus, two methods for finding the energy in a signal.

- Use $x(t) \rightarrow$ time domain
- $X(f) \rightarrow$ frequency representation

$\mathcal{G}_x(f) = \mathcal{F}[R_x(\tau)] = |X(f)|^2 \rightarrow$ energy spectral density of the signal $x(t)$

$\mathcal{G}_x(f)$ represents the amount of energy per hertz of bandwidth present in the signal at various frequencies.

2.3.2. Power-Type Signals

Time-average autocorrelation function of the power-type signal $x(t)$:

$$R_x(\tau) = \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} x(t)x^*(t-\tau)\mathrm{dt}$$

Power content of the signal can be obtained from

$$\begin{aligned} P_x &= \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} |x(t)|^2\mathrm{dt} \\ &= R_x(0) \end{aligned}$$

Define $\mathcal{S}_x(f)$ (power-spectral density) $\rightarrow \mathcal{S}_x(f) = \mathcal{F}[R_x(\tau)]$

2.3.2. Power-Type Signals

Can express power content of signal $x(t)$ in terms of $S_x(f)$ by noting:

$$\begin{aligned} R_x(0) &= \int_{-\infty}^{\infty} \mathcal{S}_x(f)\mathrm{df} \\ \therefore P_x &= R_x(0) \\ &= \int_{-\infty}^{\infty} S_x(f)\mathrm{df} \end{aligned}$$

2.3.2. Power-Type Signals

If power-type signal $x(t)$ is passed through filter with impulse response $h(t)$, the output is

$$y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)\mathrm{d}\tau$$

Time-average autocorrelation function for output:

$$R_y(\tau) = \lim_{T\to\infty} \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} y(t)y^*(t-\tau)\mathrm{dt}$$

2.3.2. Power-Type Signals

Substituting for $y(t)$

$$R_y(\tau) = \lim_{T\to\infty} \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} \left[\int_{\infty}^{-\infty} h(u)x(t-u)\mathrm{du}\right] \left[\int_{-\infty}^{\infty} h^*(v)x^*(t-\tau-v)\mathrm{d}v\right] \mathrm{dt}$$

2.3.2. Power-Type Signals

Change of variables $w = t - u$

$$\begin{aligned} R_y(\tau) &= \int_{-\infty}^{\infty} \int_{\infty}^{-\infty} h(u)h^*(v) \\ &\quad \times \lim_{T\to\infty} \frac{1}{T} \int_{-\frac{T}{2}-u}^{\frac{T}{2}+u} \left[x(w)x^*(u+w-\tau-v)\mathrm{d}w\right] \mathrm{du\ dv} \\ &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} R_x(\tau+v-u)h(u)h^*(v)\mathrm{d}u\,\mathrm{d}v \\ &= \int_{-\infty}^{\infty} \left[R_x(\tau+v) \star h(\tau+v)\right] h^*(v)\mathrm{d}v \\ &= R_x(\tau) \star h(\tau) \star h^*(-t) \end{aligned}$$

2.3.2. Power-Type Signals

Take Fourier transform on both sides

$$\begin{aligned} S_y(f) &= S_x(f)H(f)H^*(f) \\ &= S_x(f)|H(f)|^2 \end{aligned}$$

2.3.2. Power-Type Signals

For periodic signals $\rightarrow$ time-average autocorrelation function + power spectral density simplify

assume $x(t)$ periodic (period T_0 + Fourier series coefficients $\{x_n\}$)

$$\begin{aligned}
R_x(\tau) &= \lim_{T\rightarrow\infty} \frac{1}{T}\int_{-T/2}^{T/2} x(t)x^*(t-\tau)\mathrm{dt} \\
&= \lim_{k\rightarrow\infty} \frac{1}{kT_0}\int_{-kT_0/2}^{kT_0/2} x(t)x^*(t-\tau)\mathrm{dt} \\
&= \lim_{T\rightarrow\infty} \frac{k}{kT_0}\int_{-T_0/2}^{T_0/2} x(t)x^*(t-\tau)\mathrm{dt} \\
&= \tfrac{1}{T_0}\int_{-T_0/2}^{T_0/2} x(t)x^*(t-\tau)\mathrm{dt}
\end{aligned}$$

2.3.2. Power-Type Signals

Substitute the Fourier series expansion of periodic signal:

$$R_x(\tau) = \frac{1}{T_0}\int_{-T_0/2}^{T_0/2} \sum_{n=-\infty}^{\infty}\sum_{m=-\infty}^{\infty} x_n x_m^* e^{j2\pi \frac{m}{T_0}\tau} e^{j2\pi \frac{n-m}{T_0}\tau} \mathrm{dt}$$

Using fact that

$$\frac{1}{T_0}\int_{-T_0/2}^{T_0/2} e^{j2\pi \frac{n-m}{T_0}} \mathrm{dt} = \delta_{m,n}$$

We obtain

$$R_x(\tau) = \sum_{n=-\infty}^{\infty} |x_n|^2 e^{j2\pi \frac{n}{T_0} t}$$

2.3.2. Power-Type Signals

$$R_x(\tau) = \sum_{n=-\infty}^{\infty} |x_n|^2 e^{j2\pi \frac{n}{T_0} t}$$

Time-average autocorrelation function of periodic signal is itself periodic, with same period as original signals

Fourier series coefficients $\rightarrow$ magnitude squares of Fourier coefficients of original signal

2.3.2. Power-Type Signals

Power-spectral density $\rightarrow$ Fourier transform of $R_x(\tau)$

Dealing with periodic function $\rightarrow$ Fourier transform consists of impulses in frequency domain.

$$S_x(f) = \sum_{n=-\infty}^{\infty} |x_n|^2 \delta(f - \frac{n}{T_0})$$

To find power content, integrate over whole frequency spectrum

$$P_x = \sum_{n=-\infty}^{\infty} |x_n|^2$$