Pulse-Amplitude Modulation

ELEN 3024 - Communication Fundamentals

School of Electrical and Information Engineering, University of the Witwatersrand

July 15, 2013

Pulse-Amplitude Modulation

Barry *et al.*, "Digital Communication", Chapter 5

5. Introduction

bit-streams inherently discrete-time, all physical media are continuous-time in nature

modulation $\rightarrow$ bit stream represented as a continuous-time signalling

Consider PAM:

- baseband PAM
- passband transmission

5. Introduction (Continued)

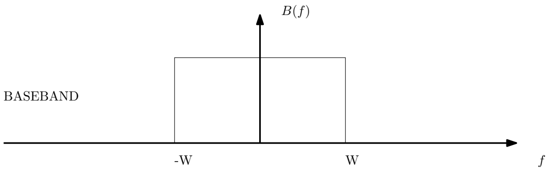


Figure: Baseband

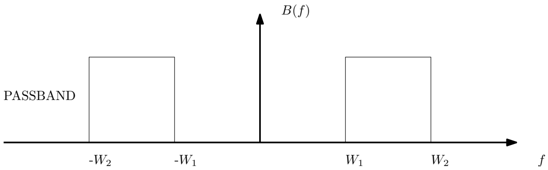


Figure: passband

5. Introduction (Continued)

Examples of PAM:

- PSK
- AM-PM
- QAM

5.1. Baseband PAM

Baseband PAM transmitter sends information by modulating the amplitudes of a series of pulses:

$$s(t) = \sum_{m=-\infty}^{\infty} a_m g(t - mT) \qquad (1)$$

$\frac{1}{T} \rightarrow$ symbol rate

$g(t) \rightarrow$ pulse shape

Set of amplitudes $\{a_m\} \rightarrow$ symbols

Signal $\rightarrow$ sequence of possibly overlapping pulses $\rightarrow$ amplitude of m'th pulse determined by m'th symbol

Equation (1) $\rightarrow$ PAM, regardless of shape of $g(t)$

5.1. Baseband PAM

Example 5.1 Concepts:

1. Mapper $\rightarrow$ converts input bit stream to modulating symbol stream

a. In practice, symbols restricted to finite alphabet $\mathcal{A}$

b. Convenient when $|\mathcal{A}| = 2^b$

2. transmit filter with impulse response $g(t)$

5.1. Baseband PAM

Note difference between baud rate (symbol rate) and bit rate

Assumption $\rightarrow$ symbols from mapper independent and identically distributed, white discrete random process

5.1.1. Nyquist Pulse Shapes

Receiver $\rightarrow$ recover transmitted symbols from a continuous-time PAM signal distorted by noisy channel

Assume for now noiseless PAM, in order to explore relationship between **bandwidth and symbol rate**

To recover the symbols $\{a_m\}$ from $s(t)$ $\rightarrow$ sample $s(t)$ at multiples of the symbol period

k-th sample:

$$\begin{aligned} s(kT) &= \sum_{m=-\infty}^{\infty} a_m g(kT - mT) \\ &= a_m * g(kT) \end{aligned}$$

Interpretation $\rightarrow$ discrete-time convolution of the symbol sequence with a sampled version of the pulse shape

5.1.1. Nyquist Pulse Shapes

Decomposing the convolution sum into two parts:

$$s(kT) = g(0)a_k + \sum_{m \neq k} a_m g(kT - mT)$$

- First term $\rightarrow$ desired signal
- Second term $\rightarrow$ intersymbol interference (ISI)

ISI $\rightarrow$ interference from neighboring symbols

5.1.1. Nyquist Pulse Shapes

When is no ISI present **OR** $s(kT) = a_k$?

5.1.1. Nyquist Pulse Shapes

When is no ISI present **OR** $s(kT) = a_k$?

When second term $\sum_{m \neq k} a_m g(kT - mT) = 0$

Alternatively:

$$g(kT) = \delta_k$$

5.1.1. Nyquist Pulse Shapes

$$g(kT) = \delta_k \tag{2}$$

Taking Fourier transform on both sides and making use of sampling theorem:

$$\frac{1}{T}\sum_{m=-\infty}^{\infty} G\left(f - \frac{m}{T}\right) = 1 \tag{3}$$

Equation 3 $\rightarrow$ Nyquist criterion

Nyquist pulse $\rightarrow$ satisfies Eq 3 (and Eq 2)

5.1.1. Nyquist Pulse Shapes

Nyquist criterion is the key that ties symbol rate to bandwidth

Nyquist criterion implies existence of a minimum bandwidth for transmitting at a certain symbol rate with no ISI

Alternatively, given certain bandwidth, maximum symbol rate for avoiding ISI.

5.1.1. Nyquist Pulse Shapes

Example 5-4

Sketch

Plot depicts $\frac{1}{T}\sum_{m=-\infty}^{\infty} G\left(f - \frac{m}{T}\right)$ for a particular pulse shape $g(t)$ whose bandwidth is less than $1/(2T)$

Effect of sampling $\rightarrow$ place an image of $G(f)$ at each multiple of the sampling rate.

Regardless of shape of $G(f)$ $\rightarrow$ always gap between images whenever the pulse shape bandwidth is less than half the symbol rate

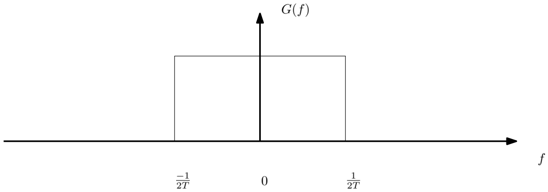
Such gaps prevent the images from adding to a constant

5.1.1. Nyquist Pulse Shapes

From example 5-4 evident minimum bandwidth required to avoid ISI is half the symbol rate $1/(2T)$

Bandwidth of $1/(2T)$ eliminates gap between aliases

in order to ensure aliases add to a constant, each alias must itself have a rectangular shape, giving $G(F)$:



5.1.1. Nyquist Pulse Shapes

Taking inverse Fourier transform → minimum-bandwidth pulse satisfying Nyquist criterion:

$$g(t) = \frac{\sin(\pi t/T)}{\pi t/T}$$

Refer to sketch

Observe that pulse has zero crossings at all multiples of T except at $t = 0$, where $g(0) = 1$

5.1.1. Nyquist Pulse Shapes

Example 5-5.

Using sinc pulses, transmit $a_0 = 1$ and $a_1 = 2$

Sketch resulting signal

5.1.1. Nyquist Pulse Shapes

Nyquist criterion implies a maximum symbol rate for a given bandwidth

If we are constrained to frequencies $|f| < W$, the maximum symbol rate that can be achieved with zero ISI is $1/T = 2W$

5.1.1. Nyquist Pulse Shapes

Minimum bandwidth is desirable, but the ideal bandlimited pulse is impractical

The bandwidth W of a practical pulse is larger than its minimum value by a factor $1 + \alpha$:

$$W = \frac{1 + \alpha}{2T}$$

$\alpha \rightarrow$ excess-bandwidth parameter

5.1.1. Nyquist Pulse Shapes

Excess bandwidth also expressed as percentage, 100 % $\rightarrow \alpha = 1$ and bandwidth of $1/T$ (twice the minimum bandwidth)

Practical systems, excess bandwidth in range of 10 % to 100 %

Increasing the excess bandwidth simplifies the implementation (simpler filtering and timing recovery) at expense of channel bandwidth

5.1.1. Nyquist Pulse Shapes

Zero-excess-bandwidth pulse is unique $\rightarrow$ ideal bandlimited pulses

non-zero excess bandwidth, pulse shape no longer unique

Commonly used pulses with nonzero excess bandwidth that satisfy the Nyquist criterion are the raised-cosine pulses, given by

$$g(t) = \left(\frac{\sin(\pi t/T)}{\pi t/T}\right)\left(\frac{\cos(\alpha\pi t/T)}{1-(2\alpha t/T)^2}\right)$$

5.1.1. Nyquist Pulse Shapes

Fourier transforms of raised-cosine pulses:

$$G(f) = \begin{cases} T, & |f| \leq \frac{1-\alpha}{2T} \\ T cos^2 \left[\frac{\pi T}{2\alpha} \left(|f| - \frac{1-\alpha}{2T} \right) \right], & \frac{1-\alpha}{2T} < |f| \leq \frac{1+\alpha}{2T} \\ 0, & \frac{1+\alpha}{2T} < |f| \end{cases}$$

5.1.1. Nyquist Pulse Shapes

Refer to Fig 5.2

$\alpha = 0 \rightarrow$ ideally bandlimited pulses

Other values of α, energy rolls of more gradually (α also roll-off factor)

Shape of roll-off is that of a cosine raised above abscissa.

5.1.2. The Impact of Filtering on PAM

Consider impact of a channel

Many important channels modeled as a linear time-invariant filter with impulse response $b(t)$ and additive noise $n(t)$

5.1.2. The Impact of Filtering on PAM

PAM signal applied to linear channel with impulse response $b(t)$ and additive noise $n(t)$:

$$r(t) = \int_{-\infty}^{\infty} b(\tau) \sum_{m=-\infty}^{\infty} a_m g(t - mT - \tau) d\tau + n(t)$$

rewritten as

$$r(t) = \sum_{m=-\infty}^{\infty} a_m h(t - mT) + n(t)$$

where $h(t) = g(t) * b(t)$ is the convolution of $g(t)$ with $b(t)$:

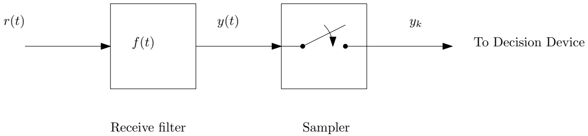
$$h(t) = \int_{-\infty}^{\infty} g(\tau) b(t - \tau) d\tau$$

5.1.2. The Impact of Filtering on PAM

$r(t) \rightarrow$ received pulse $\rightarrow$ also PAM if transmitted pulse is PAM:

- Different pulse shape
- added noise

Typical receiver front end consists of a receive filter $f(t)$ followed by a sampler



5.1.2. The Impact of Filtering on PAM

Receive filter perform several functions, including:

- compensating for the distortion of the channel
- diminishing the effect of additive noise

Receive filter conditions the received signal before sampling

If bandwidth of additive noise wider than that of transmitted signal, receive filter can reject out-of-band noise

Receive filter might be chosen to avoid ISI after sampling

5.1.2. The Impact of Filtering on PAM

Output of receive filter (input to sampler):

$$y(t) = \sum_{m=-\infty}^{\infty} a_m p(t - mT) + n'(t)$$

where $p(t) = g(t) * b(t) * f(t) \rightarrow$ overall pulse shape

noise $n'(t)$ is filtered version of the received noise $n(t)$

Receive filter output is another PAM signal, pulse shape $p(t)$ and with added noise

5.1.2. The Impact of Filtering on PAM

To avoid ISI, overall pulse shape $p(t) = g(t) * b(t) * f(t)$ must be Nyquist

Thus, $p(kT) = \delta_k$, or $\sum_m P(f - \frac{m}{T}) = T$

When this condition is satisfied, the k-th sample $y(kT)$ reduces to a_k plus noise, with no interference from $\{a_{l \neq k}\}$

Since $P(f) = G(f)B(f)F(f)$, a bandwidth limitation on the channel necessarily leads to the same bandwidth limitation on the overall pulse shape.

Thus, it is the channel bandwidth W that determines the maximum symbol rate, namely $1/T = 2W$

5.1.3. ISI and Eye diagrams

Self study

5.1.4. Bit rate and Spectral Efficiency

Symbols independent and uniform from alphabet $\mathcal{A}$ of size $|\mathcal{A}| \rightarrow$ each symbol conveys $\log_2|\mathcal{A}|$ bits of information

Transmits $1/T$ symbols per second, bit rate is:

$$R_b = \frac{\log_2|\mathcal{A}|}{T} \text{b/s}$$

5.1.4. Bit rate and Spectral Efficiency

Increase bit rate

- Increase size of alphabet
- increase symbol rate

Symbol rate is bounded by the bandwidth constraints of channel

Size of alphabet constrained by:

- allowable transmitted power
- severity of the additive noise on the channel

5.1.4. Bit rate and Spectral Efficiency

Constraints on symbol rate and alphabet size limits available bit rate for a given channels

Spectral efficiency:

$$\nu = \frac{R_b}{W}$$

5.1.4. Bit rate and Spectral Efficiency

Baseband PAM:

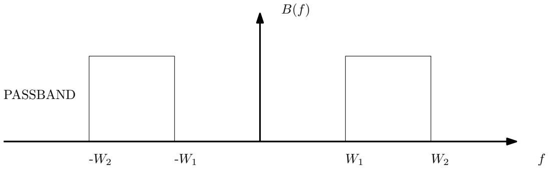
$$\nu = \frac{R_b}{W} = \frac{\log_2|\mathcal{A}|/T}{(1+\alpha)/(2T)} = \frac{2\log_2|\mathcal{A}|}{1+\alpha}$$

Maximal spectral efficiency:

$$\nu_{max} = 2\log_2|\mathcal{A}|$$

5.2. Passband PAM

Many practical communication channels are passband in nature $\rightarrow$ frequency response that of bandpass filtered



5.2.1. Three Representations of Passband PAM

<u>Method 1</u>

Start with suboptimal strategy $\rightarrow$ pulse-amplitude-modulation double-sideband

Passband channel has bandwidth B

Start with a real-valued baseband PAM signal with bandwidth $B/2$

Modulate carrier frequency f_c, by multiplying f_c and baseband PAM:

$$s(t) = \sqrt{2}\cos(2\pi f_c t) \sum_k a_k g(t - kT)$$

5.2.1. Three Representations of Passband PAM

Method 1

Modulated signal will pass undistorted through channel when pulse shape $g(t)$ is low-pass with bandwidth $B/2$

Avoiding ISI, symbol rate is twice the pulse shape bandwidth (Symbol rate $= 1/T = B$)

Maximal spectral efficiency of PAM-DSB with real alphabet $\mathcal{A}$ is $\log_2|\mathcal{A}|$

5.2.1. Three Representations of Passband PAM

Method 2

Recognise upper sideband and lower sideband of $s(t)$ conveys identical information

Double spectral efficiency by using single-sideband (SSB), transmit only one sideband

Disadvantage $\rightarrow$ difficulty in realizing filtering

5.2.1. Three Representations of Passband PAM

Method 3

Recognise that PAM-DSB carries information only in in-phase component

Quadrature component is zero

Double the spectral efficiency of PAM-DSB by transmitting a second baseband PAM signal in quadrature:

$$s(t) = \sqrt{2}\cos(2\pi f_c t)\sum_k a_k^I g(t-kT) - \sqrt{2}\sin(2\pi f_c t)\sum_k a_k^Q g(t-kT)$$

5.2.1. Three Representations of Passband PAM

Method 3

Bandwidth the same as PAM-DSB, but conveys twice as much information

Assume both baseband PAM signals use the same pulse shape

Symbols modulating in-phase and quadrature components are denoted $\{a_k^I\}$ and $\{a_k^Q\}$

QAM $\rightarrow$ $\{a_k^I\}$ and $\{a_k^Q\}$ chosen independently from same real alphabet $\mathcal{A}$

See Fig 5.10

5.2.1. Three Representations of Passband PAM

Complex Envelope

Can represent $s(t)$ in terms of complex envelope:

$$s(t) = \sqrt{2}\text{Re}\left\{\widetilde{s}(t)e^{j2\pi f_c t}\right\}$$

where the complex envelope of a passband PAM signal is:

$$\widetilde{s}(t) = \sum_k a_k g(t - kT)$$

with $a_k = a_k^I + ja_k^Q$

5.2.1. Three Representations of Passband PAM

Complex Envelope

Observe that complex envelope of passband PAM looks exactly like real-valued baseband PAM signal

Passband PAM signal $\rightarrow$ signal whose complex envelope is the baseband PAM signal with complex symbols and a real pulse shape

For a realization, refer to Fig 5-11 (Theoretical)

5.2.1. Three Representations of Passband PAM

Comparing Method 2 and Method 3

Both passband PAM and PAM-SSB double spectral efficiency of PAM-DSB

PAM-SSB $\rightarrow$ fixes the bit rate but cuts bandwidth in half

passband PAM $\rightarrow$ doubles bit rate while keeping the bandwidth fixed

5.2.1. Three Representations of Passband PAM

Another representation for Passband PAM

Another presentation of passband PAM $\rightarrow$ express data symbols a_m in polar coordinates

$$a_m = c_m e^{j\theta_m}$$

So that

$$\begin{aligned} s(t) &= \sqrt{2}\text{Re}\left\{\sum_{-\infty}^{\infty} c_m e^{j2\pi f_c t + \theta_m} g(t - mT)\right\} \\ &= \sqrt{2}\sum_{-\infty}^{\infty} c_m \cos(2\pi f_c t + \theta_m) g(t - mT) \end{aligned}$$

Each pulse $g(t - mT)$ multiplied by carrier, where amplitude and phase of the carrier is determined by the amplitude and phase of a_m

Sometimes called AM/PM

5.2.2. Constellations

Alphabet $\rightarrow$ set $\mathcal{A}$ of symbols available for transmission

Baseband signal has real-valued alphabet

Passband PAM signal $\rightarrow$ alphabet that is a set of complex numbers

For real-valued and complex alphabets $\rightarrow$ each symbol represents $\log_2|\mathcal{A}|$ bits

5.2.2. Constellations

Complex-valued alphabet is best described by plotting the alphabet as a set of points in a complex plain

Plot $\rightarrow$ signal constellation

Example 5-12 - on blackboard

Example 5-13 - on blackboard

5.2.2. Constellations

Energy of alphabet:

Assumptions:

- All symbols are equally likely
- pulse shape is normalized to have unit energy

Expected energy E of a single passband PAM pulse transmitted in isolation,

$$s(t) = \sqrt{2}\text{Re}\left\{\widetilde{s}(t)e^{j2\pi f_c t}\right\}$$

with $\widetilde{s}(t) = ag(t)$:

$$\begin{aligned} E &= E[\int_{-\infty}^{\infty} s^2(t)dt] \\ &= E[\int_{-\infty}^{\infty} |\widetilde{s}(t)|^2 dt] \\ &= E[|a|^2] \int_{-\infty}^{\infty} g(t)^2 dt \\ &= \frac{1}{|\mathcal{A}|} \sum_{a \in \mathcal{A}} |a|^2 \end{aligned}$$

5.2.2. Constellations

Power: Leave output

5.2.2. Constellations

Alphabet Design

Distance between points in a constellation determines the likelihood that one point will be confused with another

Minimum distance d_{min} between two points key parameter of the constellation

Two constellations can be considered to have the approximately the same noise immunity if the minimum distance d_{min} is the same.

To make d_{min} the same for constellations with different number of points, higher point constellations require more transmit power

Either a power or an error-probability penalty associated with using larger constellations

5.2.2. Constellations

Objective of signal constellation design $\rightarrow$ maximize distance between symbols while not exceeding power constraint

Optimal constellations difficult to derive or costly to implement

5.2.2. Constellations

Assume average power constraint

Performance of a constellation depends only on distances among symbols $\rightarrow$ performance of constellation invariant under translation

Should translate a constellation so that its power is minimized

Power minimized if it has zero mean

5.2.2. Constellations

Given a set of symbols $\{a_i\}$, translate with complex number m such that the power

$$E[|a - m|^2] = \sum_{i=1}^{M} p_a(a_i)|a_i - m|^2$$

of translated symbol set $\{a_i - m\}$ is minimized

Best choice for translation: $m = E[a]$

Proof:

for any other transformation n

$$\begin{aligned} E[|a-n|^2] &= E[|(a-m)+(m-n)|^2] \\ &= E[|a-m|^2] + 2\mathrm{Re}\{(m-n)^*(E[a]-m)\} + |m-n|^2 \\ &= E[|a-m|^2] + |m-n|^2 \end{aligned}$$

Mean energy under translation n is larger than mean energy under translation m by $|m-n|^2$

5.2.2. Constellations

Problem of optimal design of constellation is complicated

5.2.2. Constellations

QAM $\rightarrow$ Square $\rightarrow$ $M = 2^b$, b even $\rightarrow$ Fig. 5-13

QAM $\rightarrow$ $M = 2^b$, b odd $\rightarrow$ Fig. 5-14

PSK and PSK + amplitude modulation $\rightarrow$ Fig. 5-15

Hexagonal constellations $\rightarrow$ Fig 5-16 (Hexagonal refer to shape of decision regions)

5.2.2. Constellations

PSK

2-PSK $\rightarrow$ binary phase-shift keying (BPSK)

4-PSK $\rightarrow$ QPSK / 4-QAM

M elements of M-PSK:

$$a = \sqrt{E} e^{j2\pi m/M}, \text{ for } m \in \{0, \ldots, M-1\}$$

Pure PSK $\rightarrow$ constant envelope $\rightarrow$ robust against amplifier nonlinearities

5.2.3. Spectral Efficiency

Bit rate $R_b \to \log_2|\mathcal{A}| \times \frac{1}{T}$

Spectral efficiency $\nu = R_b/\text{bandwidth}$

Difference between baseband PAM and passband PAM is relationship between symbol rate and bandwidth

Passband $\to$ bandwidth W $\to$ maximal symbol rate is W

Due to bandwidth of passband PAM signal being twice bandwidth of pulse shape (upconversion process)

$$\nu = \frac{R_b}{W} = \frac{\log_2|\mathcal{A}|}{1+\alpha}$$

5.2.3. Spectral Efficiency

Passband PAM lower spectral efficiency than baseband PAM baseband

Complex alphabet (Passband) is much larger than real alphabet (baseband)

If using QAM and transmit L levels on each of the two quadrature carriers:

$$\nu = \log_2 L^2 = 2 \cdot \log_2 L \text{ bits / sec-Hz}$$

Same as for baseband PAM with L levels