Digital Transmission through the Additive White Gaussian Noise Channel

ELEN 3024 - Communication Fundamentals

School of Electrical and Information Engineering, University of the Witwatersrand

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Digital Transmission Through the AWGN Channel

Proakis and Salehi, “Communication Systems Engineering” (2nd Ed.), Chapter 7
Overview
Convert output of a signal source into a sequence of binary digits

Now consider transmission of digital information sequence over communication channels characterized as additive white Gaussian noise channels

AWGN channel → one of the simplest mathematical models for various physical communication channels

Most channels are analog channels → digital information sequence mapped into analog signal waveforms
Introduction

Focus on:

- characterization, and
- design

of analog signal waveforms that carry digital information and performance on an AWGN channels

Consider both baseband and passband signals.

Baseband $\rightarrow$ no need for carrier

passband channel $\rightarrow$ information-bearing signal impressed on a sinusoidal carrier
Gram-Schmidt orthogonalization → construct an orthonormal basis for a set of signals

Develop a geometric representation of signal waveforms as points in a signal space

Representation provides a compact characterization of signal sets, simplifies analysis of performance

Using vector representation, waveform communication channels are represented by vector channels (reduce complexity of analysis)
Suppose set of $M$ signal waveforms $s_m(t)$, $1 \leq m \leq M$ to be used for transmitting information over comms channel.

From set of $M$ waveforms, construct set of $N \leq M$ orthonormal waveforms $\rightarrow N$ dimension of signal space.

Use Gram-Schmidt orthogonalization procedure.
7.1.1. Gram-Schmidt Orthogonalization Procedure

Given first waveform $s_1(t)$, with energy $\mathcal{E}_1 \rightarrow$ first waveform of the orthonormal set:

$$\psi_1(t) = \frac{s_1(t)}{\sqrt{\mathcal{E}_1}}$$
Second waveform → constructed from $s_2(t)$ by computing the projection of $s_2(t)$ onto $\psi_1(t)$:

$$c_{21} = \int_{-\infty}^{\infty} s_2(t)\psi_1(t)dt$$

Then, $c_{21}\psi_1(t)$ is subtracted from $s_2(t)$ to yield:

$$d_2(t) = s_2(t) - c_{21}\psi_1(t)$$
7.1.1. Gram-Schmidt Orthogonalization Procedure

\(d_2(t)\) is orthogonal to \(\psi_1\), but energy of \(d_2(t) \neq 1\).

\[
\psi_2(t) = \frac{d_2(t)}{\sqrt{\mathcal{E}_2}}
\]

\[
\mathcal{E}_2 = \int_{-\infty}^{\infty} d_2^2(t) dt
\]
7.1.1. Gram-Schmidt Orthogonalization Procedure

In general, the orthogonalization of the \( k \)th function leads to

\[
\psi_k(t) = \frac{d_k(t)}{\sqrt{\mathcal{E}_k}}
\]

where

\[
d_k(t) = s_k(t) - \sum_{i=1}^{k-1} c_{ki} \psi_i(t)
\]

\[
\mathcal{E}_k = \int_{-\infty}^{\infty} d_k^2(t) dt
\]

and

\[
c_{ki} = \int_{-\infty}^{\infty} s_k(t) \psi_i(t) dt, \ i = 1, 2, \ldots, k - 1
\]
Orthogonalization process is continued until all the $M$ signal waveforms $\{s_m(t)\}$ have been exhausted and $N \leq M$ orthonormal waveforms have been constructed.

The $N$ orthonormal waveforms $\{\psi_n(t)\}$ forms a basis in the $N$-dimensional signal space.

Dimensionality $N = M$ if all signal waveforms are linearly independent.
Example 7.1.1

Selfstudy
7.1.1. Gram-Schmidt Orthogonalization Procedure

Can express the $M$ signals $\{s_m(t)\}$ as exact linear combinations of the $\{\psi_n(t)\}$

$$s_m(t) = \sum_{n=1}^{N} s_{mn} \psi_n(t), \ m = 1, 2, \ldots, M$$

where

$$s_{mn} = \int_{-\infty}^{\infty} s_m(t) \psi_n(t) dt$$

$$\mathcal{E}_m = \int_{-\infty}^{\infty} s_m^2(t) dt = \sum_{n=1}^{N} s_{mn}^2$$

Thus

$$s_m = \left( s_{m1}, s_{m2}, \ldots, s_{mN} \right)$$
Energy of the $m$th signal → square of length of vector or square of Euclidean distance from origin to point in $N$-dimensional space.

Inner product of two signals equal to inner product of their vector representations

$$\int_{-\infty}^{\infty} s_m(t)s_n(t)dt = s_m \cdot s_n$$

Thus, any $N$-dimensional signal can be represented geometrically as a point in the signal space spanned by the $N$ orthonormal functions $\{\psi_n(t)\}$.
7.1.1. Gram-Schmidt Orthogonalization Procedure

Example 7.1.2

Selfstudy
Set of basis functions \( \{\psi_n(t)\} \) obtained by Gram-Schmidt procedure is not unique.
7.2. Pulse Amplitude Modulation

Pulse Amplitude Modulation → information conveyed by the amplitude of the transmitted signal
7.2.1. Baseband Signals

Binary PAM → simplest digital modulation method

Binary 1 → pulse with amplitude $A$

Binary 0 → pulse with amplitude $-A$

Also referred to as binary antipodal signalling

Pulses transmitted at a bit rate $R_b = 1/T_b$ bits/sec ($T_b$ → bit interval)
7.2.1. Baseband Signals

Generalization of PAM to nonbinary (M-ary) pulse transmission straightforward

Instead of transmitting one bit at a time, binary information sequence is subdivided into blocks of \( k \) bits → symbol

Each symbol represented by one of \( M = 2^k \) pulse amplitude values

\( k = 2 \rightarrow M = 4 \) pulse amplitude values

When bitrate \( R_b \) is fixed, symbol interval

\[
T = \frac{k}{R_b} = kT_b
\]
7.2.1. Baseband Signals

In general $M$-ary PAM signal waveforms may be expressed as

$$s_m(t) = A_m g_T(t), \quad m = 1, 2, \ldots, M, \quad 0 \leq t \leq T$$

where $g_T(t)$ is a pulse of some arbitrary shape (example $\rightarrow$ Fig. 7.7.)

Distinguishing feature among the $M$ signals is the signal amplitude

All the $M$ signals have the same pulse shape
Another important feature → energies

\[ \mathcal{E}_m = \int_0^T s_m^2(t)dt = A_m^2 \int_0^T g_{T}^2(t)dt = A_m^2 \mathcal{E}_g, \quad m = 1, 2, \ldots, M \]

\( \mathcal{E}_g \) is the energy of the signal pulse \( g_T(t) \)
7.2.2. Bandpass Signals

To transmit digital waveforms through a bandpass channel by amplitude modulation, the baseband signal waveforms $s_m(t), \ m = 1, 2, \ldots, M$ are multiplied by a sinusoidal carrier of the form $\cos 2\pi f_c t$
7.2.2. Bandpass Signals

Transmitted signal waveforms:

\[ u_m(t) = A_m g_T(t) \cos 2\pi f_c t, \quad m = 1, 2, \ldots, M \]

Amplitude modulation \( \rightarrow \) shifts the spectrum of the baseband signal by an amount \( f_c \rightarrow \) places signal into passband of the channel

Fourier transform of carrier: \( \left[ \delta(f - f_c) + \delta(f + f_c) \right] / 2 \)
7.2.2. Bandpass Signals

Spectrum of amplitude-modulated signal

\[ U_m(t) = \frac{A_m}{2} [G_T(f - f_c) + G_T(f + f_c)] \]

Spectrum of baseband signal \( s_m(t) = A_m g_T(t) \) is shifted in frequency by amount \( f_c \)

Result \( \rightarrow \) DSB-SC AM \( \rightarrow \) Fig. 7.9

Upper sideband \( \rightarrow \) frequency content of \( u_m(t) \) for
\( f_c < |f| \leq f_c + W \)

Lower sideband \( \rightarrow \) frequency content of \( u_m(t) \) for
\( f_c - W \leq |f| < f_c \)

\( u_m(t) \rightarrow \) bandwidth \( = 2W \) \( \rightarrow \) twice bandwidth of baseband signal
7.2.2. Bandpass Signals

Energy of bandpass signal waveforms \( u_m(t), \ m = 1, 2, \ldots, M \)

\[
\mathcal{E}_m = \int_{-\infty}^{\infty} u_m^2(t) \, dt \\
= \int_{-\infty}^{\infty} A_m^2 g_T^2(t) \cos^2 2\pi f_c t \, dt \\
= \frac{A_m^2}{2} \int_{-\infty}^{\infty} g_T^2(t) \, dt + \frac{A_m^2}{2} \int_{-\infty}^{\infty} g_T^2(t) \cos 4\pi f_c t \, dt
\]

When \( f_c \gg W \)

\[
\int_{-\infty}^{\infty} g_T^2(t) \cos 4\pi f_c t \, dt = 0
\]

Thus,

\[
\mathcal{E}_m = \frac{A_m^2}{2} \int_{-\infty}^{\infty} g_T^2(t) = \frac{A_m^2}{2} \mathcal{E}_g
\]
7.2.2. Bandpass Signals

\( E_g \rightarrow \) energy in the signal \( g_T(t) \)

Energy in bandpass signal is one-half of the energy of the baseband signal

Assume \( g_T(t) \)

\[
g_T(T) = \begin{cases} 
\sqrt{\frac{E_g}{T}} & 0 \leq t < T \\
0, & \text{otherwise}
\end{cases}
\]

\( \Rightarrow \) amplitude-shift keying (ASK)
7.2.3. Geometric Representation of PAM Signals

Baseband signals for $M$-ary PAM $\rightarrow s_m(t) = a_m g_T(t), M = 2^k$, $g_T(t)$ pulse with peak amplitude normalized to unity

$M$-ary PAM waveforms are one-dimensional signals, expressed as

$$s_m(t) = s_m \psi(t), \ m = 1, 2, \ldots, M$$

basis function $\psi(t)$

$$\psi(t) = \frac{1}{\sqrt{\mathcal{E}_g}} g_T(t), \ 0 \leq t \leq T$$

$\mathcal{E}_g \rightarrow$ energy of signal pulse $g_T(t)$
7.2.3. Geometric Representation of PAM Signals

signal coefficients $\rightarrow$ one-dimensional vectors

$$s_m = \sqrt{E_g} A_m, \quad m = 1, 2, \ldots, M$$

Important parameter $\rightarrow$ Euclidean distance between two signal points:

$$d_{mn} = \sqrt{|s_m - s_n|^2} = \sqrt{E_g (A_m - A_n)^2}$$

$\{A_m\}$ symmetrically spaced about zero and equally distant between adjacent signal amplitudes $\rightarrow$ symmetric PAM

Refer to Fig 7.11
7.2.3. Geometric Representation of PAM Signals

PAM signals have different energies.

Energy of $m$th signal

$$\mathcal{E}_m = s_m^2 = \mathcal{E}_g A_m^2, \quad m = 1, 2, \ldots, M$$

Equally probable signals, average energy is given as:

$$\mathcal{E}_{av} = \frac{1}{M} \sum_{m=1}^{M} \mathcal{E}_m = \frac{\mathcal{E}_g}{M} \sum_{m=1}^{M} A_m^2$$
7.2.3. Geometric Representation of PAM Signals

If signal amplitudes are symmetric about origin

\[ A_m = (2m - 1 - M), \quad m = 1, 2, \ldots, M \]

Average energy

\[ E_{av} = \frac{E_g}{M} \sum_{m=1}^{M} (2m - 1 - M)^2 = E_g (M^2 - 1) / 3 \]
7.2.3. Geometric Representation of PAM Signals

When baseband PAM impressed on a carrier, basic geometric representation of the digital PAM signal waveforms remain the same.

Bandpass signal waveforms $u_m(t)$ expressed as

$$u_m(t) = s_m \psi(t)$$

where

$$\psi(t) = \sqrt{\frac{2}{E_g}} g_T(t) \cos 2\pi f_c t$$

and

$$s_m = \sqrt{\frac{E_g}{2}} A_m, \quad m = 1, 2, \ldots, M$$
7.3. Two-Dimensional Signal Waveforms

PAM signal waveforms are basically one-dimensional signals.

Now consider the construction of two-dimensional signals.
7.3.1 Baseband Signals

Two signal waveforms \(s_1(t)\) and \(s_2(t)\) orthogonal over interval \((0, T)\) if

\[
\int_0^T s_1(t)s_2(t)dt = 0
\]

Fig. 7.12 → two examples

\[
\mathcal{E} = \int_0^T s_1^2(t)dt = \int_0^T s_2^2(t)dt = \int_0^T [s_1']^2(t)dt = \int_0^T [s_2']^2(t)dt = A^2 T
\]

Either pair of these signals may be used to transmit binary information, one signal waveform → 1, the other waveform → 0
7.3.1 Baseband Signals

Geometrically, signal waveforms represented as signal vectors in two-dimensional space

One choice, select unit energy, rectangular functions

\[ \psi_1(t) = \begin{cases} \sqrt{2/T}, & 0 \leq t \leq T/2 \\ 0, & \text{otherwise} \end{cases} \]

\[ \psi_2(t) = \begin{cases} \sqrt{2/T}, & T/2 < t \leq T \\ 0, & \text{otherwise} \end{cases} \]
7.3.1 Baseband Signals

Signal waveforms $s_1(t)$ and $s_2(t)$ expressed as

$$s_1(t) = s_{11} \psi_1(t) + s_{12} \psi_2(t)$$
$$s_2(t) = s_{21} \psi_2(t) + s_{22} \psi_2 t$$

where

$$s_1 = (s_{11}, s_{12}) = \left(A \sqrt{T/2}, A \sqrt{T/2}\right)$$
$$s_2 = (s_{21}, s_{22}) = \left(A \sqrt{T/2}, -A \sqrt{T/2}\right)$$

Fig 7.13 → plot of $s_1$ and $s_2$

Signals are separated by $90^\circ$ → orthogonal
7.3.1 Baseband Signals

Square of length of each vector gives the energy in each signal

\[
\mathcal{E}_1 = ||s_1||^2 = A^2 T \\
\mathcal{E}_2 = ||s_2||^2 = A^2 T
\]

Euclidean distance between two signals is

\[
d_{12} = \sqrt{||s_1 - s_2||^2} = A\sqrt{2T} = \sqrt{2A^2 T} = \sqrt{2\mathcal{E}}
\]

\[
\mathcal{E}_1 = \mathcal{E}_2 = \mathcal{E} \rightarrow \text{signal energy}
\]
7.3.1 Baseband Signals

Similarly:

\[ s_1' = (A\sqrt{T}, 0) = (\sqrt{E}, 0) \]
\[ s_2' = (0, A\sqrt{T}) = (0, \sqrt{E}) \]

Euclidean distance between \( s_1' \) and \( s_2' \) identical to that of \( s_1 \) and \( s_2 \)
7.3.1 Baseband Signals

Suppose we wish to construct four signal waveforms in two dimensions

Four signal waveforms → transmit 2 bits in signalling interval of length $T$

use $-s_1$ and $-s_2$

Obtain 4-point signal constellation → Fig. 7.15

$s_1(t)$ and $s_2(t)$ orthogonal, plus $-s_1(t)$ and $-s_2(t)$ orthogonal → biorthogonal signals
Procedure for constructing a larger set of signal waveforms relatively straightforward

add additional signal points (signal vectors) in two-dimensional plane, construct corresponding waveforms by using the two orthonormal basis functions $\psi_1(t)$ and $\psi_2(t)$

Suppose construct $M = 8$ two-dimensional signal waveforms, all of equal energy $E$.

Fig. 7.16 → constellation diagram

Transmit 3 bits at a time
7.3.1 Baseband Signals

Remove condition that all 8 waveforms have equal energy

Example: select 4 biorthogonal waveforms with energy $E_1$ and another 4 biorthogonal waveforms with energy $E_2$ ($E_2 > E_1$)

Refer to Fig. 7.17
7.3.2 Two-dimensional Bandpass Signals - Carrier-Phase Modulation

Bandpass PAM $\rightarrow$ set of baseband signals impressed on carrier

Similarly, set of $M$ two-dimensional signal waveforms $s_m(t)$, $m = 1, 2, \ldots, M$ create a set of bandpass signal waveforms

$$u_m(t) = s_m(t) \cos 2\pi f_c t, \quad m = 1, 2, \ldots, M, \quad 0 \leq t \leq T$$
Consider special case in which $M$ two-dimensional bandpass signal waveforms constrained to have same energy:

\[
\mathcal{E}_m = \int_0^T u_m^2(t) \, dt = \int_0^T s_m^2(t) \cos^2 2\pi f_c t \, dt = \frac{1}{2} \int_0^T s_m^2(t) \, dt + \frac{1}{2} \int_0^T s_m^2(t) \cos 4\pi f_c t \, dt = \frac{1}{2} \int_0^T s_m^2(t) \, dt = \mathcal{E}_s, \text{ for all } m
\]

When all signal waveforms have same energy, corresponding signal points fall on circle with radius $\sqrt{\mathcal{E}_s}$

Fig. 7.15 → example of constellation with $M = 4$
7.3.2 Two-dimensional Bandpass Signals - Carrier-Phase Modulation

Signal points equivalent to a single signal whose phase is shifted → carrier-phase modulated signal

\[ u_m(t) = g_T(t) \cos\left(2\pi f_c t + \frac{2\pi m}{M}\right), \quad M = 0, 1, \ldots, M - 1, \]
7.3.2 Two-dimensional Bandpass Signals - Carrier-Phase Modulation

When \( g_T(t) \) rectangular pulse

\[
g_T(t) = \sqrt{\frac{2\mathcal{E}_s}{T}}, \quad 0 \leq t \leq T
\]

Corresponding transmitted signal waveforms

\[
u_m(t) = \sqrt{\frac{2\mathcal{E}_s}{T}} \cos \left( 2\pi f_c t + \frac{2\pi m}{M} \right),
\]

has constant envelope, carrier phase changes abruptly at beginning of each signal interval

\( \Rightarrow \) phase-shift keyeing (PSK)

Fig 7.18. QPSK signal waveform
7.3.2 Two-dimensional Bandpass Signals - Carrier-Phase Modulation

Can rewrite carrier-phase modulated signal equation as

\[ u_m(t) = g_T(t)A_{mc} \cos 2\pi f_c t - g_T(t)A_{ms} \sin 2\pi f_c t \]

where

\[ A_{mc} = \cos 2\pi m/M \]
\[ A_{ms} = \sin 2\pi m/M \]

Phase-modulated signal may be viewed as two quadrature carriers with amplitudes \( g_T(t)A_{mc} \) and \( g_T(t)A_{ms} \) (Fig. 7.19)
7.3.2 Two-dimensional Bandpass Signals - Carrier-Phase Modulation

Thus, digital phase-modulated signals can be represented geometrically as two-dimensional vectors

\[ s_m = (\sqrt{\mathcal{E}_s} \cos \frac{2\pi m}{M}, \sqrt{\mathcal{E}_s} \sin \frac{2\pi m}{M}) \]

Orthogonal basis functions are

\[ \psi_1(t) = \sqrt{\frac{2}{\mathcal{E}_g}} g_T(t) \cos 2\pi f_c t \]
\[ \psi_2(t) = -\sqrt{\frac{2}{\mathcal{E}_g}} g_T(t) \sin 2\pi f_c t \]

Fig. 7.20 → signal point constellations for \( M = 2,4,8 \)
7.3.2 Two-dimensional Bandpass Signals - Carrier-Phase Modulation

Mapping or assignment of $k$ information bits into the $M = 2^k$ possible changes may be done in number of ways

Preferred mapping $\rightarrow$ Gray encoding (Fig. 7.20)

Most likely errors caused by noise $\rightarrow$ selection of an adjacent phase to transmitted phase $\rightarrow$ single bit error
7.3.2 Two-dimensional Bandpass Signals - Carrier-Phase Modulation

Euclidean distance between any two signal points in constellation

\[ d_{mn} = \sqrt{||s_m - s_n||^2} = \sqrt{2E_s \left(1 - \cos \frac{2\pi(m-n)}{M}\right)} \]

Minimum Euclidean distance (distance between two adjacent signal points)

\[ d_{min} = \sqrt{2E_s \left(1 - \cos \frac{2\pi}{M}\right)} \]

\[ d_{min} \rightarrow \text{determine error-rate performance of receiver in AWGN} \]
7.3.3 Two-dimensional Bandpass Signals - Quadrature Amplitude Modulation

When $E_s$ not equal for every symbol, we can impress separate information “bits” on each of the quadrature carriers ($\cos 2\pi f_c t$ and $\sin 2\pi f_c t$) → Quadrature Amplitude Modulation (QAM)

Form of quadrature-carrier multiplexing

$$u_m(t) = A_{mc}g_T(t)\cos 2\pi f_c t + A_{ms}g_T(t)\sin 2\pi f_c t, \quad m = 1, 2, \ldots, M$$

$\{A_{mc}\}$ and $\{A_{ms}\}$ are the sets of amplitude levels obtained by mapping $k$-bit sequences into signal amplitudes.
7.3.3 Two-dimensional Bandpass Signals - Quadrature Amplitude Modulation

Fig. 7.21 → 16-QAM → amplitude modulating each quadrature carrier by $M = 4$ PAM

QAM → combined digital-amplitude and digital-phase modulation

$$u_{mn}(t) = A_m g_T(t) \cos(2\pi f_c t + \theta_n), \quad m = 1, 2, \ldots, M_1,$$
$$n = 1, 2, \ldots, M_2$$

If $M_1 = 2^{k_1}$ and $M_2 = 2^{k_2}$ →
7.3.3 Two-dimensional Bandpass Signals - Quadrature Amplitude Modulation

Fig. 7.21 → 16-QAM → amplitude modulating each quadrature carrier by $M = 4$ PAM

QAM → combined digital-amplitude and digital-phase modulation

$$u_{mn}(t) = A_m g_T(t) \cos(2\pi f_c t + \theta_n), \quad m = 1, 2, \ldots, M_1,$$
$$n = 1, 2, \ldots, M_2$$

If $M_1 = 2^{k_1}$ and $M_2 = 2^{k_2}$ → $k_1 + k_2 = \log_2(M_1 \times M_2)$ bits, at symbol rate $R_b/(k_1 + k_2)$
7.3.3 Two-dimensional Bandpass Signals - Quadrature Amplitude Modulation

Fig. 7.22. → Functional block diagram of modulator for QAM
7.3.3 Two-dimensional Bandpass Signals - Quadrature Amplitude Modulation

Geometric signal representation of the signals:

\[ s_m = \left( \sqrt{\mathcal{E}_s A_{mc}}, \sqrt{\mathcal{E}_s A_{ms}} \right) \]

Fig. 7.23 → Examples of signal space constellations for QAM.

Average transmitted energy → sum of the average energies on each of the quadrature carriers
7.3.3 Two-dimensional Bandpass Signals - Quadrature Amplitude Modulation

For rectangular signal constellations, average energy/symbol

$$\mathcal{E}_{av} = \frac{1}{M} \sum_{i=1}^{M} \| s_i \|^2$$
7.3.3 Two-dimensional Bandpass Signals - Quadrature Amplitude Modulation

Euclidean distance

\[ d_{mn} = \sqrt{||s_m - s_n||^2} \]
7.3.3 Two-dimensional Bandpass Signals - Quadrature Amplitude Modulation