Forward Error Correction

1 Linear Block Codes

1. Determine all the codewords of the (n, k) linear code C with parity-check matrix

$$H = \begin{bmatrix} 1 & 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 \end{bmatrix}.$$

   Determine the parameters n, k and d_{min} of this code.

2. Consider an (8,4) linear systematic code C with parity-check equations

$$\begin{aligned} \overline{v}_0 &= u_1 + u_2 + u_3 \\ \overline{v}_1 &= u_0 + u_1 + u_2 \\ \overline{v}_2 &= u_0 + u_1 + u_3 \\ \overline{v}_3 &= u_0 + u_2 + u_3 \end{aligned}$$

   where (u_0, u_1, u_2, u_3) forms the message and v_0, v_1, v_2 and v_3 are the parity-check bits.

   - Determine a generator and parity matrix for C.
   - Show that $d_{min} = 4$.
   - Determine the syndrome equations in terms of the received vector $\overline{r}$.

3. Let C be the (6, 3) linear code with generator matrix

$$G = \begin{bmatrix} 1 & 0 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

   - Encode the message (0 1 1).
   - Determine a parity-check matrix for C.
   - Determine a standard array for C.
   - What is the d_{min} of C?
   - Is C a Hamming code?
   - Decode the received vector $\overline{r} = (1\ 0\ 0\ 1\ 1\ 0)$.

4. Let C be a linear code with parity-check matrix

$$H = \begin{bmatrix} 1 & 1 & 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 1 & 0 & 0 \end{bmatrix}$$

   Assume the vector received is (1 1 0 1 0 1) and that only one error occurred during transmission. Find the transmitted codeword.

5. Determine a parity-check matrix for a Hamming code of length 15. Is the vector (0 1 1 0 1 0 1 1 0 1 1 1 0 0 0) a valid codeword?

2 Cyclic Codes

1. Consider the (3,2) cyclic code generated by $1 + x$.
   (a) Tabulate all the messages with their corresponding code vectors as well as code polynomials
   (b) Is the above code in systematic form?

2. Consider the (7,3) cyclic code generated by $1 + x + x^2 + x^4$.
   (a) Tabulate all the messages with their corresponding code vectors as well as code polynomials
   (b) Determine the systematic generator matrix of the code.

3. Consider the (7,4) cyclic systematic code generated by $1 + x + x^3$. Encode the message $u(x) = 1 + x^3$ in systematic form.

4. Consider the (15,11) cyclic systematic Hamming code C generated by the primitive polynomial $p(x) = 1 + x + x^4$.
   (a) Determine the syndrome of $r(x) = 1 + x^2 + x^7$.
   (b) Determine the d_{min}, n and k of the (n, k) cyclic code generated by $(x + 1)(1 + x + x^4)$.
   (c) Determine the parity polynomial $h(x)$ of C.
   (d) Determine the generator polynomial of C_d.
   (e) Determine the generator and parity-check matrix (both in systematic form) of C.
   (f) Tabulate the possible error polynomials and corresponding syndromes.

5. Consider the polynomial $g(x) = 1 + x^2 + x^4 + x^6 + x^7 + x^{10}$.
   (a) Show that $g(x)$ generates a (21,11) cyclic code.
   (b) Determine the syndrome of $r(x) = 1 + x^5 + x^{17}$.

3 Reed-Solomon Codes

1. Construct the Galois field $GF(2^3)$ with primitive polynomial $x^3 + x + 1$ and state each element's binary and power representation.

2. Determine the generator polynomial of a Reed-Solomon code of length 7 that can correct 2 errors. The polynomial $(x - \alpha^1)$ must be a factor of $g(x)$. Use the Galois field constructed in Question 1. Write down the parameters of this code.

3. Determine if $\alpha^6 + \alpha^2 x + \alpha^3 x^2 + \alpha^4 x^3 + \alpha^6 x^4 + \alpha^4 x^5 + \alpha^6 x^6$ is a code polynomial of $C = < g(x) >$.

4. Define an erasure. How do erasures differ from normal errors?

5. How many errors and erasures can a Reed-Solomon code correct?