Angle Modulation

ELEN 3024 - Communication Fundamentals

School of Electrical and Information Engineering, University of the Witwatersrand

July 15, 2013

Angle Modulation

Proakis and Salehi, “Communication Systems Engineering” (2nd Ed.), Chapter 3

Overview

3.3.Angle Modulation

Amplitude-modulation methods → linear-modulation methods (AM DSB-FC not linear)

FM and PM other analogue modulation techniques.

FM → frequency of carrier f_c changed by message

PM → phase of carrier is changed by variations in message signal

FM and PM → angle-modulation methods → nonlinear

3.3.Angle Modulation

Angle-modulation $\rightarrow$ due to nonlinearity

- complex to implement
- Difficult to analyse

Many cases only approximate analysis.

Bandwidth-expansion of angle modulation $\rightarrow$ effective bandwidth of modulated signal $>>$ bandwidth of message signal

$\Rightarrow$ Trade-off bandwidth for high noise immunity

3.3.1. Representation of FM and PM signals

Angle-modulated signal:

$$u(t) = A_c \cos(\theta(t))$$

$\theta(t) \rightarrow$ phase of the signal

Instantaneous frequency $f_i(t)$:

$$f_i(t) = \frac{1}{2\pi} \frac{d}{dt} \theta(t)$$

3.3.1. Representation of FM and PM signals

Since $u(t)$ bandpass signal:

$$u(t) = A_c \cos(2\pi f_c t + \phi(t))$$

Therefore,

$$f_i(t) =$$

3.3.1. Representation of FM and PM signals

Since $u(t)$ bandpass signal:

$$u(t) = A_c \cos(2\pi f_c t + \phi(t))$$

Therefore,

$$f_i(t) = f_c + \frac{1}{2\pi}\frac{d}{dt}\phi(t)$$

3.3.1. Representation of FM and PM signals

PM $\rightarrow$ message $m(t) \rightarrow \phi(t) = k_p m(t)$

FM:

$$f_i(t) - f_c = k_f m(t) = \frac{1}{2\pi} \frac{d}{dt} \phi(t)$$

k_p and k_f $\rightarrow$ phase and frequency deviation constants

$$\phi(t) = \begin{cases} k_p m(t), & \text{PM} \\ 2\pi k_f \int_{-\infty}^{t} m(\tau)\, d\tau, & \text{FM} \end{cases}$$

3.3.1. Representation of FM and PM signals

Observations:

FM $\rightarrow$ phase modulate carrier with integral of a message.

Or

$$\frac{d}{dt}\phi(t) = \begin{cases} k_p \frac{d}{dt} m(t), & \text{PM} \\ 2\pi m(t), & \text{FM} \end{cases}$$

PM $\rightarrow$ frequency modulate carrier with derivative of message $m(t)$

Fig 3.25: Important.

Fig 3.26: Important.

3.3.1. Representation of FM and PM signals

Demodulation of FM signal $\rightarrow$ finding instantaneous frequency of the modulated signal and subtracting the carrier frequency from it.

Demodulation of PM signal $\rightarrow$ finding the phase of the signal and then recovering $m(t)$

Maximum phase deviation in PM system $\rightarrow$
$\Delta\phi_{max} = k_p \max\left[|m(t)|\right]$

Maximum frequency-deviation in FM $\rightarrow$ $\Delta f_{max} = k_f \max\left[|m(t)|\right]$

Example 3.3.1.

Message signal $\rightarrow$ $m(t) = a\cos(2\pi f_m t)$

Modulate FM system and PM system

Find the modulated signal in each case.

For PM

$$\phi(t) =$$

Example 3.3.1.

Message signal $\rightarrow$ $m(t) = a\cos(2\pi f_m t)$

Modulate FM system and PM system

Find the modulated signal in each case.

For PM

$$\phi(t) = k_p m(t) = k_p a \cos(2\pi f_m t)$$

Example 3.3.1.

Message signal $\rightarrow m(t) = a\cos(2\pi f_m t)$

Modulate FM system and PM system

Find the modulated signal in each case.

For PM

$$\phi(t) = k_p m(t) = k_p a \cos(2\pi f_m t)$$

For FM

$$\phi(t) =$$

Example 3.3.1.

Message signal $\rightarrow m(t) = a\cos(2\pi f_m t)$

Modulate FM system and PM system

Find the modulated signal in each case.

For PM

$$\phi(t) = k_p m(t) = k_p a\cos(2\pi f_m t)$$

For FM

$$\phi(t) = 2\pi k_f \int_{-\infty}^{t} m(\tau)\, d\tau = \frac{k_f a}{f_m}\sin(2\pi f_m t)$$

Example 3.3.1.

Modulated signals:

$$u(t) = \left\{ \right.$$

Example 3.3.1.

Modulated signals:

$$u(t) = \begin{cases} A_c \cos(2\pi f_c t + k_p a \cos(2\pi f_m t)), & \text{PM} \\ A_c \cos(2\pi f_c t + \frac{k_f a}{f_m} \sin(2\pi f_m t)), & \text{FM} \end{cases}$$

Define

$\beta_p = k_p a$ and $\beta_f = \frac{k_f a}{f_m}$

we have

$$u(t) = \begin{cases} A_c \cos(2\pi f_c t + \beta_p \cos(2\pi f_m t)), & \text{PM} \\ A_c \cos(2\pi f_c t + \beta_f \sin(2\pi f_m t)), & \text{FM} \end{cases}$$

β_p and β_f $\rightarrow$ modulation indices

3.3.1. Representation of FM and PM signals

We can extend the definition of the modulation index for a general nonsinusoidal signal $m(t)$ as

$$\beta_p = k_p \max\left[|m(t)|\right]$$

$$\beta_f = \frac{k_f \max\left[|m(t)|\right]}{W}$$

In terms of the maximum phase and frequency deviation:

$$\beta_p = \Delta\phi_{max}$$

$$\beta_f = \frac{\Delta f_{max}}{W}$$

3.3.1.1 Narrowband Angle Modulation

If k_p or k_f and $m(t)$ such that $\phi(t) \ll 1 \;\; \forall t$:

$$\begin{aligned} u(t) &= A_c \cos(2\pi f_c t)\cos(\phi(t)) - A_c \sin(2\pi f_c t)\sin(\phi(t)) \\ &\approx A_c \cos(2\pi f_c t) - A_c \phi(t)\sin(2\pi f_c t) \end{aligned}$$

Modulated signal very similar to conventional AM signal (AM DSB FC)

Sine wave modulated by $m(t)$ instead of cosine

Bandwidth $\approx$

3.3.1.1 Narrowband Angle Modulation

If k_p or k_f and $m(t)$ such that $\phi(t) \ll 1 \;\; \forall t$:

$$\begin{aligned} u(t) &= A_c \cos(2\pi f_c t) \cos(\phi(t)) - A_c \sin(2\pi f_c t) \sin(\phi(t)) \\ &\approx A_c \cos(2\pi f_c t) - A_c \phi(t) \sin(2\pi f_c t) \end{aligned}$$

Modulated signal very similar to conventional AM signal (AM DSB FC)

Sine wave modulated by $m(t)$ instead of cosine

Bandwidth $\approx$ Bandwidth(AM) $\rightarrow 2 \times$ bandwidth($m(t)$)

3.3.1.1 Narrowband Angle Modulation

Fig. 3.27 $\rightarrow$ phasor diagrams for narrowband angle modulation and AM

Narrowband angle modulation far less amplitude variations than AM

Narrowband angle modulation $\rightarrow$ constant amplitude

Slight amplitude variations due to approximation

Narrowband angle-modulation does not provide better noise immunity compared to AM DSB FC.

3.3.2 Spectral Characteristics of Angle-Modulated Signals

Due to inherent nonlinearity of angle-modulation $\rightarrow$ difficult to characterise spectral properties

Study simple modulation signals and certain approximations

Generalized to more complicated messages

Study 3 cases for $m(t)$:

- sinusoidal signal
- periodic signal
- nonperiodic signal

3.3.2.1 Angle Modulation by a Sinusoidal Signal

For both PM and FM

$$u(t) = A_c \cos(2\pi f_c t + \beta \sin(2\pi f_m t))$$

$\beta \rightarrow$ modulation index

$$u(t) = Re\left(A_c e^{j2\pi f_c t} e^{j\beta sin(2\pi f_m t)}\right)$$

3.3.2.1 Angle Modulation by a Sinusoidal Signal

Since $\sin(2\pi f_m t)$ periodic with period $T_m = \frac{1}{f_m}$, same true for complex exponential signal $e^{j\beta \sin(2\pi f_m t)}$

Therefore, can be expanded in Fourier series representation:

$$\begin{aligned} c_n &= f_m \int_0^{\frac{1}{f_m}} e^{j\beta \sin(2\pi f_m t)} e^{-jn2\pi f_m t} \, dt \\ &= \frac{1}{2\pi} \int_0^{2\pi} e^{j\beta \sin u - nu} \, du \quad (u = 2\pi f_m t) \end{aligned}$$

Last integral $\rightarrow$ *Bessel function of the first kind of order n* $\rightarrow J_n(\beta)$

3.3.2.1 Angle Modulation by a Sinusoidal Signal

Therefore, Fourier series for complex exponential

$$e^{j\beta \sin 2\pi f_m t} = \sum_{n=-\infty}^{\infty} J_n(\beta) e^{j2\pi n f_m t}$$

Substituting into complex baseband representation

$$\begin{aligned} u(t) &= \text{Re}\left(A_c \sum_{n=-\infty}^{\infty} J_n(\beta) e^{j2\pi n f_m t} e^{j2\pi f_c t}\right) \\ &= \sum_{n=-\infty}^{\infty} A_c J_n(\beta) \cos(2\pi(f_c + n f_m)t) \end{aligned}$$

Even for single sinusoidal modulating signal, angle-modulated signal contains all frequencies of the form $f_c + nf_m$ for $n = 0, \pm 1, \pm 2, \ldots$

Actual bandwidth → infinite

3.3.2.1 Angle Modulation by a Sinusoidal Signal

Amplitude of sinusoidal components of frequencies $f_c + nf_m$, n large $\rightarrow$ very small

Therefore define finite effective bandwidth of modulated wave

Series expansion of Bessel function:

$$J_n(\beta) = \sum_{k=0}^{\infty} \frac{(-1)^k \left(\frac{\beta}{2}\right)^{n+2k}}{k!(k+n)!}$$

3.3.2.1 Angle Modulation by a Sinusoidal Signal

For small β, can use following approximation

$$J_n(\beta) \approx \frac{\beta^n}{2^n n!}$$

Thus for small $\beta \rightarrow$ only first sideband corresponding to $n = 1$ of importance

3.3.2.1 Angle Modulation by a Sinusoidal Signal

Properties of Bessel function (verified by expansion):

$$J_{-n}(\beta) = \begin{cases} J_n(\beta), & n \text{ even} \\ -J_n(\beta), & n \text{ odd} \end{cases}$$

Fig. 3.28 $\rightarrow$ Plots of $J_n(\beta)$ for various values of n

Table 3.1. $\rightarrow$ Table of the values of the Bessel function

Example 3.3.2

carrier $\rightarrow$ $c(t) = 10\cos(2\pi f_c t)$

message $\rightarrow$ $\cos(20\pi t)$

message used to frequency modulate carrier with $k_f = 50$

Find expression for the modulated signal and determine how many harmonics should be selected to contain 99% of the modulated signal power

3.3.2.1 Angle Modulation by a Sinusoidal Signal

In general, effective bandwidth of an angle-modulated signal which contains at least 98 % of the signal power:

$$B_c = 2(\beta + 1) f_m$$

$\beta \rightarrow$ modulation index

f_m frequency of sinusoidal message signal.

3.3.2.1 Angle Modulation by a Sinusoidal Signal

Consider effect of amplitude and frequency of sinusoidal $m(t)$ on bandwidth and number of harmonics in modulated signal

$$m(t) = a\cos(2\pi f_m t)$$

bandwidth (effective) is given by:

$$B_c = 2\left(\beta + 1\right) f_m = \begin{cases} 2(k_p a + 1)f_m, & \text{PM} \\ 2\left(\frac{k_f a}{f_m} + 1\right) f_m, & \text{FM} \end{cases}$$

or,

$$B_c = \begin{cases} 2(k_p a + 1)f_m, & \text{PM} \\ 2(k_f a + f_m), & \text{FM} \end{cases}$$

3.3.2.1 Angle Modulation by a Sinusoidal Signal

Increasing $a \rightarrow$ in PM and FM almost same effect on increasing bandwidth B_c

Increasing f_m:

- PM $\rightarrow$ increase in B_c is proportional to increase in f_m
- FM $\rightarrow$ increase in B_c is additive (for large β not substantial)

3.3.2.1 Angle Modulation by a Sinusoidal Signal

Consider Harmonics:

$$M_c = 2\lfloor \beta \rfloor + 3 = \begin{cases} 2\lfloor k_p a \rfloor + 3, & \text{PM} \\ 2\left\lfloor \frac{k_f a}{f_m} \right\rfloor + 3, & \text{FM} \end{cases}$$

Increasing $a \rightarrow$ increases the number of harmonics

Increasing f_m

- No effect on PM
- Almost linear decrease in number of harmonics for FM

3.3.2.2 Angle Modulation by a Periodic Message Signal

Consider periodic message signal $m(t)$

For PM

$$u(t) = A_c \cos(2\pi f_c t + \beta m(t))$$

rewrite as

$$u(t) = A_c Re\left[e^{j2\pi f_c t} e^{j\beta m(t)}\right]$$

3.3.2.2 Angle Modulation by a Periodic Message Signal

Assume $m(t)$ is periodic with period $T_m = 1/f_m \rightarrow e^{j\beta m(t)}$ periodic, same period:

$$e^{j\beta m(t)} = \sum_{n=-\infty}^{\infty} c_n e^{j2\pi n f_m t}$$

where

$$\begin{aligned} c_n &= \frac{1}{T_m} \int_{T_m}^{0} e^{j\beta m(t)} e^{-j2\pi n f_m t} dt \\ &\overset{u=2\pi f_m t}{=} \frac{1}{2\pi} \int_0^{2\pi} e^{j\left[\beta m\left(\frac{u}{2\pi f_m}\right) - nu\right]} du \end{aligned}$$

and

$$\begin{aligned} u(t) &= A_c Re\left[\sum_{n=-\infty}^{\infty} c_n e^{j2\pi f_c t} e^{j2\pi n f_m t}\right] \\ &= A_c \sum_{n=-\infty}^{\infty} |c_n| \cos(2\pi(f_c + n f_m)t + \angle c_n) \end{aligned}$$

3.3.2.2 Angle Modulation by a Periodic Message Signal

Spectral characteristics of angle-modulated signal for a general non-periodic deterministic message signal $m(t)$ quite involved

Carson's rule $\rightarrow$ approximate relation for effective bandwidth:

$$B_c = 2\,(\beta + 1)\,W$$

β is modulation index defined as

$$\beta = \begin{cases} k_p \max[|m(t)|], & \text{PM} \\ \frac{k_f \max[|m(t)|]}{W}, & \text{FM} \end{cases}$$

$W \rightarrow$ bandwidth of message signal